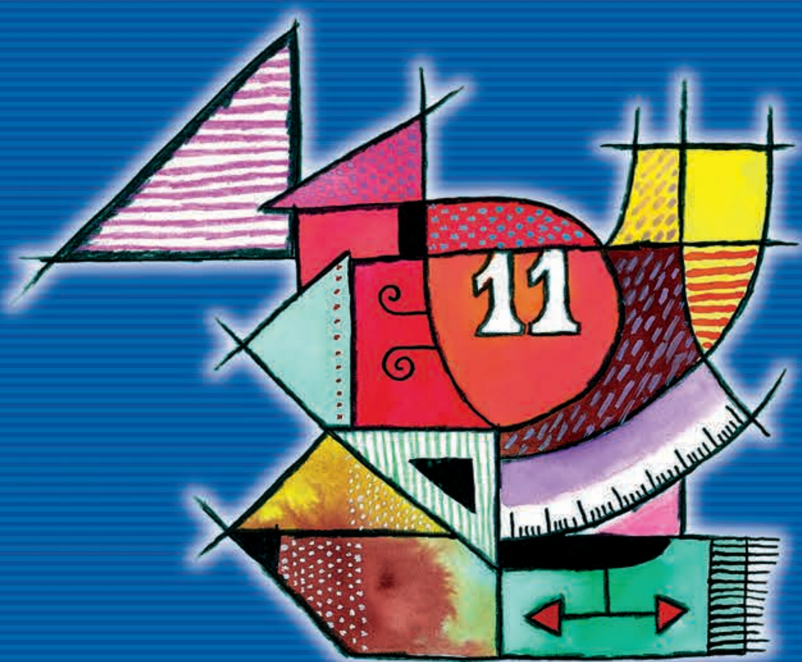
colourful

Mathematics

11



MOZAIK

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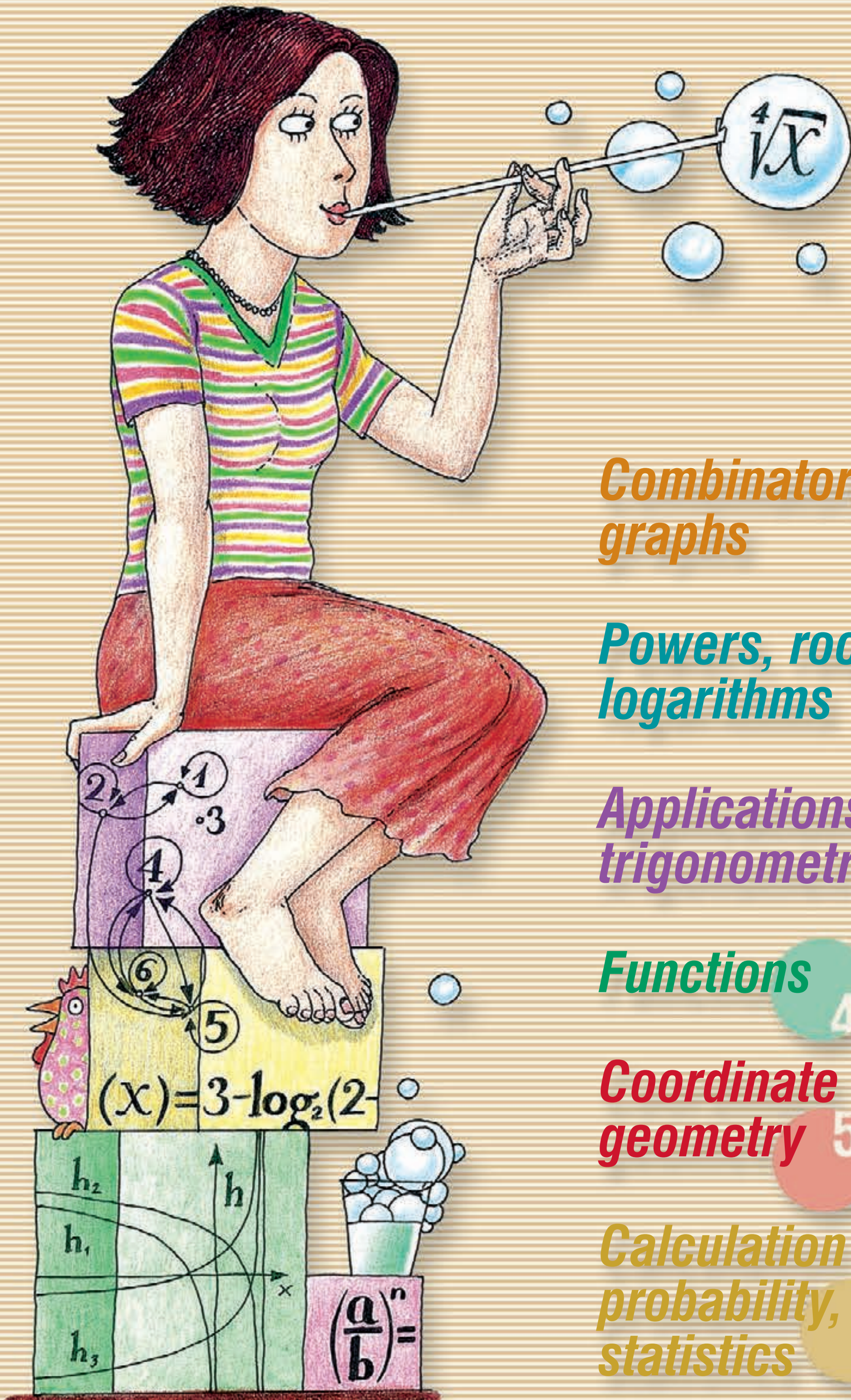
Mathematics

textbook

11

Mozaik Education – Szeged, 2015



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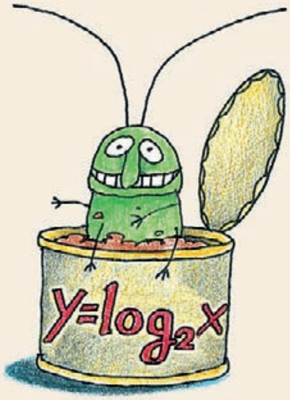
Combinatorics, graphs

Powers, roots, logarithms

Applications of trigonometry

Functions



Coordinate geometry

Calculation of probability, statistics

Guide to use the course book

The notations and highlights used in the book help with acquiring the courseware.

– The train of thought of the worked examples shows samples how to understand the methods and processes and how to solve the subsequent exercises.
– The most important definitions and theorems are denoted by colourful highlights.
– The parts of the courseware in small print and the worked examples noted in claret colour help with deeper understanding of the courseware. These pieces of knowledge are necessary for the higher level of graduation.
– Figures, the key points of the given lesson, review and explanatory parts along with interesting facts of the history of Mathematics can be found on the margin.

The difficulty level of the examples and of the appointed exercises is denoted by three different colours:

Yellow: drilling exercises with basic level difficulty; the solution and drilling of these exercises is essential for the progress.
Blue: exercises the difficulty of which corresponds to the intermediate level of graduation.
Claret: problems and exercises that help with preparing for the higher level of graduation.

These colour codes correspond to the notations used in the Colourful mathematics workbooks of Mozaik Kiadó. The workbook series contains more than 3000 exercises, which are suitable for drilling, working on in lessons and which help with preparing for the graduation.

◆ ◆ ◆

Mathematics, “ratio” and logic way of thinking are probably the most efficient tools of cognition of our world, which sometimes associate with unexplainable phenomena. These are inseparable from Homo sapiens and these make the everyday activities complete.

A few thoughts from those who have experienced all this:

“Everyone should be endowed with the skill to get the hang of the phenomena, by setting eventuality aside. Every human being needs it.” (Ákos Császár, Hungarian mathematician)

“Naturally certain parts of Mathematics can be used for improving the structured thinking more efficiently than other parts. Such way of thinking has an increasing role in being able to cope with the complexity of modern life.” (Zoltán Dienes, Hungarian mathematician)

“However Mathematics is not some far-away incomprehensibility for which another brain would be needed; we approach a novel with the same (sole) brain just as Mathematics. Mathematics reports about our existence and its prosperity. We always talk about the same thing; sometimes we hear the voice of Flaubert, sometimes of Bolyai, of Pilinszky or of Gödel. If we keep our ears open.” (Péter Esterházy, Hungarian writer)

“And those, who are learning, no matter whether to become a scientist or to prepare for a practical profession, they should not rest satisfied with simply storing the material heard or read, but they should put on their critical eye-glasses, they should analyse, dissect and compare, as otherwise the indigestible knowledge collected will only be a burden.” (Frigyes Riesz, Hungarian mathematician)

The Authors wish productive work and learning.

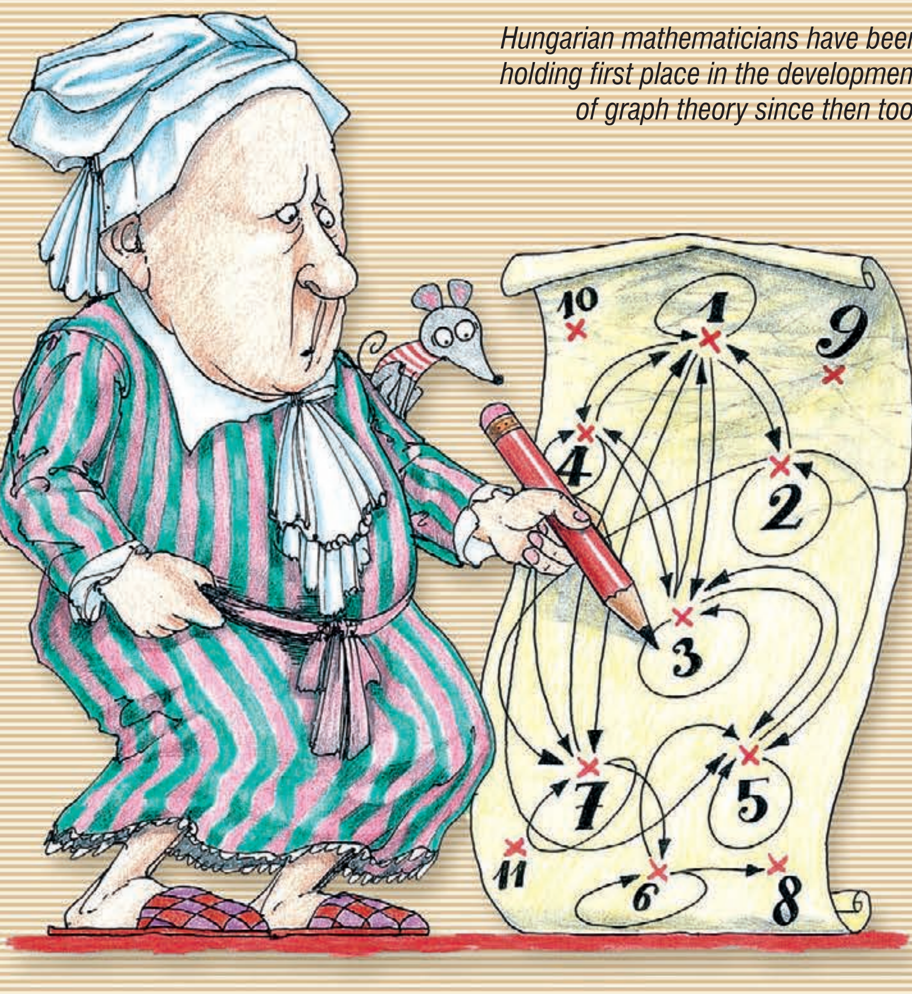
Combinatorics, graphs

The history of graph theory dates from the work of Euler (1707–1783) published in 1736, who was dealing with the problem that became famous under the name "the Konigsberg bridges".

More and more geometric problems were examined, which investigated the properties of figures independent from size and shape; these were initially called the "geometry of place" (geometria situs in Latin).

The first graph theory book of scientific quality was written by the Hungarian mathematician Dénes Kőnig (1884–1944), who was a private teacher at the Budapest University of Technology; it was published in 1936 with the title "Theorie der endlichen und unendlichen Graphen" (Theory of finite and infinite graphs).

Hungarian mathematicians have been holding first place in the development of graph theory since then too.



5. Miscellaneous counting exercises
(extra-curricular topic)

Previously we got familiar with some main types of combinatorial exercises and their solutions. In the case of particular exercises it is not easy to decide which type they are rated to, thus we have to try for the understanding application of the solution methods instead of the formulae.

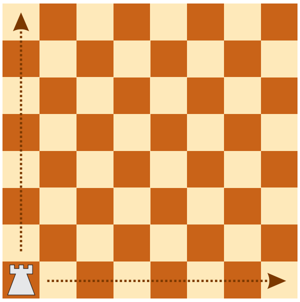


Figure 6

We can step any number of squares parallel with the sides of the chessboard with a castle (rook).

Example 1

In how many different ways can we get from the bottom left corner of the chessboard to the top right corner with a castle by moving closer to the target with every step ***(Figure 6)***:

a) in 14 steps; *b)* in 13 steps; *c)* in 12 steps?

Solution (a)

We can get from the bottom left corner to the top right corner in 14 steps, if we move one square every time, 7 times to the right and 7 times upwards. These steps can be arranged in $\frac{14!}{7!\cdot 7!} = 20\,592$ different ways, so this is the number of the possible distinct routes.

Solution (b)

We can get from the bottom left corner to the top right corner in 13 steps, if we move two squares once, and we move one square in the other steps. We can do this so that

- we move to the right 2 squares once and one square 5 times and we move upwards one square 7 times, or
- we move to the right one square 7 times and we move upwards 2 squares once and one square 5 times.

Thus the number of possibilities is: $\frac{13!}{5!\cdot 7!}\cdot 2 = 20\,592.$

Solution (c)

We can get from the bottom left corner to the top right corner in 12 steps, if

- we move to the right 3 squares once and one square four times and we move upwards one square 7 times, or $\frac{12!}{4!\cdot 7!}$
- we move upwards 3 squares once and one square four times and we move to the right one square 7 times, or $\frac{12!}{4!\cdot 7!}$
- we move to the right 2 squares once and one square 5 times and we move upwards 2 squares once and one square 5 times, or $\frac{12!}{5!\cdot 5!}$
- we move to the right 2 squares twice and one square 3 times and we move upwards one square 7 times, or $\frac{12!}{2!\cdot 3!\cdot 7!}$
- we move upwards 2 squares twice and one square 3 times and we move to the right one square 7 times. $\frac{12!}{2!\cdot 3!\cdot 7!}$

Thus the number of possibilities is:

$$\frac{12!}{4!\cdot 7!}\cdot 2 + \frac{12!}{5!\cdot 5!} + \frac{12!}{2!\cdot 3!\cdot 7!}\cdot 2 = 57\,024.$$

Example 2

In a French deck of 52 cards there are 4 suits (hearts, diamonds, spades and clubs) and 13 ranks (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A) of each suit. In how many different ways can we choose 5 cards so that the order does not matter and

a) there are no two cards of the same rank;
b) there are exactly two cards of the same rank (one pair);
c) there are exactly two pairs of cards of the same rank (two pairs);
d) there are exactly three cards of the same rank (three of a kind);
e) there are three cards of one rank and two cards of another rank (full house);
f) there are four cards of the same rank (four of a kind);
g) the ranks are adjacent ones, but their suits do not matter (straight);
h) the ranks are adjacent ones, and they are the same suit (straight flush).

Solution (a)

We can choose the 5 cards in $\binom{52}{5} = 2\,598\,960$ different ways without respect to the order. Each of the five cards can be of 4 suits, and we have to choose five out of the 13 ranks; the number of possibilities is:

$$4^5 \cdot \binom{13}{5} = 1\,317\,888.$$

The chance that there will be no two cards of the same rank is:
$\frac{1\,317\,888}{2\,598\,960} \approx 0.5.$

Solution (b)

The suit of the two alike ranks can be chosen in $\binom{4}{2}$ different ways and the suit of the other 3 distinct ranks in 4^3 different ways. We can choose the rank of the pair in 13 different ways, the other 3 ranks can be chosen out of the remaining 12, thus the number of possibilities is:

$$\binom{4}{2} \cdot 4^3 \cdot 13 \cdot \binom{12}{3} = 1\,098\,240.$$

The chance that there will be one pair is:
$\frac{1\,098\,240}{2\,598\,960} \approx 0.42.$

Solution (c)

The suits of the two pairs of alike ranks can be chosen in $\binom{4}{2} \cdot \binom{4}{2}$ different ways, and the suit of the fifth card in 4 different ways. The rank of the fifth card can be chosen after the two ranks of the two pairs of cards in 11 different ways, thus the number of possibilities is:

$$\binom{4}{2} \cdot \binom{4}{2} \cdot 4 \cdot \binom{13}{2} \cdot 11 = 123\,552.$$

The chance that there will be two pairs is:
$\frac{123\,552}{2\,598\,960} \approx 0.0475.$

Solution (d)

The suit of the three alike ranks can be chosen in $\binom{4}{3} = 4$ different ways and the suit of the other 2 distinct ranks in 4^2 different ways. We can choose the rank of the three of a kind cards in 13 different ways, the other two ranks can be chosen out of 12, thus the number of possibilities is:

$$4^3 \cdot 13 \cdot \binom{12}{2} = 54\,912.$$

The chance that there will be three of a kind is:
$\frac{54\,912}{2\,598\,960} \approx 0.02.$

Solution (e)

The suit of the three alike ranks can be chosen in $\binom{4}{3} = 4$ different ways and the suit of the two alike ranks in $\binom{4}{2}$ different ways. We can choose the rank of the three of a kind cards in 13 different ways, and the rank of the pair in 12 different ways, thus the number of possibilities is:

$$4 \cdot \binom{4}{2} \cdot 13 \cdot 12 = 3744.$$

The chance that there will be a full house is: $\frac{3744}{2\,598\,960} \approx 0.0014.$

Solution (f)

If there are four of a kind of one rank, then all 4 suits are there, it is possible in one way only; the 5th card can be any of the 4 suits. We can choose the rank of the four of a kind cards in 13 different ways, and the rank of the remaining card in 12 different ways, thus the number of possibilities is: $4 \cdot 13 \cdot 12 = 624$.

The chance that there will be four of a kind is: $\frac{624}{2\,598\,960} \approx 0.00024.$

Solution (g)

Each of the 5 cards can be of 4 suits, and the 5 adjacent ranks can be chosen in 9 different ways, thus the number of possibilities is: $4^5 \cdot 9 = 9216$.

The chance that there will be a straight is: $\frac{9216}{2\,598\,960} \approx 0.0035.$

Solution (h)

The suit of the 5 cards of the same suit can be chosen out of 4 suits, and the 5 adjacent ranks can be chosen in 9 different ways, thus the number of possibilities is: $4 \cdot 9 = 36$.

The chance that there will be a straight flush is: $\frac{36}{2\,598\,960} \approx 0.000014.$

Example 3

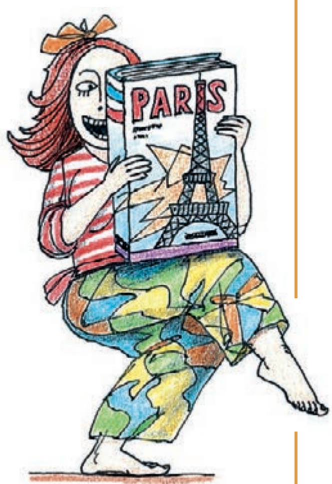
10 different books are drawn among the students of a class of 30 so that one student can win several books. In how many different ways is it possible that Sue (there is only one Sue in the class)

a) wins the Paris travel guide and nothing else?

b) wins the Paris travel guide?

c) wins exactly one book?

d) wins at least one book?



Solution (a)

Sue can win the Paris travel guide in one way only, the other 9 books can be drawn among the other 29 students; the number of possibilities is: $1 \cdot 29^9$.

Solution (b)

Sue wins the Paris travel guide, the other 9 books can be drawn among the 30 students; the number of possibilities is: 30^9.

Solution (c)

We can choose the book Sue wins in 10 different ways, the other 9 books can be drawn among 29 students; the number of possibilities is: $10 \cdot 29^9$.

Solution (d)

Sue can win at least one book if she wins 1, 2, 3, 4, 5, 6, 7, 8, 9 or 10 books; counting these cases takes a lot of time.

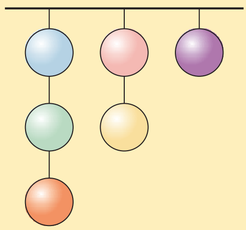
The solution is obtained faster if we subtract from all the 30^{10} possibilities those, when Sue does not win any book, and it is 29^{10} possibilities; thus she can win at least one book in a total of $30^{10} - 29^{10}$ different ways.

Exercises

1. The teacher realised that every student of the class shook hands with 6 girls and 8 boys. The number of handshakes between a boy and a girl is 5 less than the number of all the other handshakes. How many students are there in the class?
2. How many positive whole numbers are there for which exactly three of the following five characteristics hold: has two digits; is a square number; is even; is divisible by 3, but is not divisible by 9; is a prime?
3. Steven, Zack and Alan went fishing together. All three of them caught fish, which they put in their own buckets. Together they caught five fish: a meagre, a carp, a pike, a catfish and a wall eye. In how many different ways can the fish be arranged in the buckets of the three anglers? (Two arrangements are considered to be different, if there is a fish that is not in the same bucket.)
4. The big cat tamer would like to walk out 5 lions and 4 tigers into the ring, but two tigers should not follow each other. In how many different ways can he order the animals, if we distinguish the tigers and also the lions? Generalise for the case of n lions and k tigers.
5. In how many different ways can five red, three white and two blue balls be arranged so that no two white balls are next to each other? (The balls of same colour are considered to be alike.)
6. Three children picked 40 apples, then (without cutting any) they shared them.
   a) How many distributions are possible if the apples are distinguished?
   b) How many distributions are possible if the apples are not distinguished?
   c) How many distributions are possible if the apples are not distinguished, but it is restricted that each child should get at least one apple?
   d) Generalise!
7. Into at most how many parts do(es) 1, 2, 3, 4, 5 and in general n circles divide the plane?

Puzzle

Six different coloured balloons are hanging on three strings in a shooting gallery according to the figure. In how many different orders can all the balloons be shot, if one can shoot only at a balloon, which is the lowest intact one on its string?

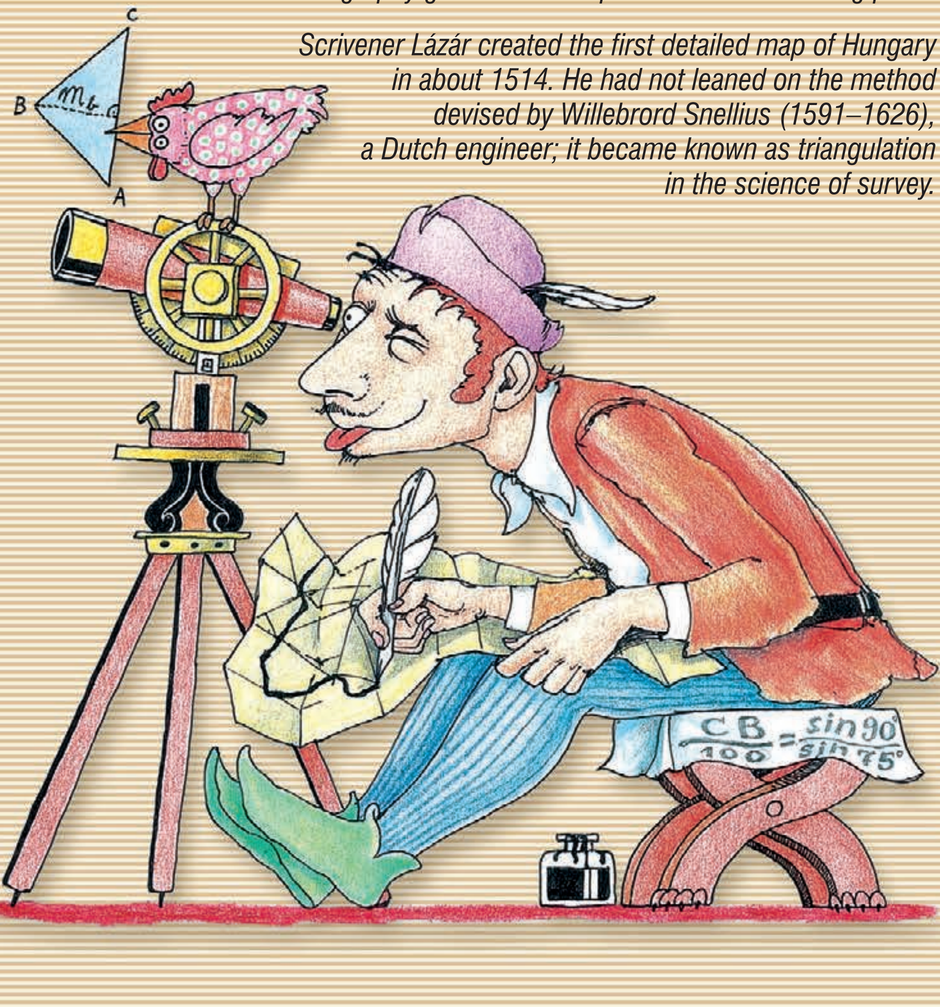


Applications of trigonometry

The world surrounding us often reveals its real face in the language of mathematics. However, often we do not understand this language because we have not yet formulated words or rules for it. But when we manage to do it, every image – which earlier was incomprehensible or incoherent – might become clear.

The first written relic, in which we can find trigonometry, is the book of Ptolemy with the title Almagest (The Great Treatise) that is mainly an astronomical work. This chapter of Mathematics was not even considered to be a separate topic. However the age of great geographical discoveries and the improvement of cartography gave the development of this area a big push.

Scrivener Lázár created the first detailed map of Hungary in about 1514. He had not leaned on the method devised by Willebrord Snellius (1591–1626), a Dutch engineer; it became known as triangulation in the science of survey.



3. Dot product in the coordinate system

We know that we can precisely mark out a star in the sky or we can unambiguously determine a place on Earth if we give the coordinates that can be assigned to the given object in the corresponding coordinate system.

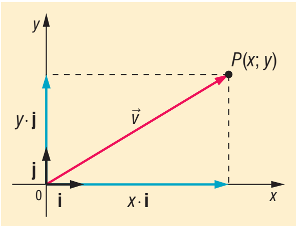


Figure 17

These positions can be given also with the help of the corresponding position vectors. However the position vectors can be also expressed by the *linear combination of the base vectors* of the coordinate system, and these base vectors can unambiguously be given with the help of the coordinates corresponding to the point. ***(Figure 17)***

According to this an arbitrary position vector can be given in terms of the base vectors in the following form:

$$\vec{v} = x\mathbf{i} + y\mathbf{j}.$$

The one-to-one correspondence implies that both the position vector and the point marked out by it can be described with the coordinate pair $(x; y)$. It can be expressed with notations in the following form:

$$P(x; y) \quad \text{or} \quad \vec{v}(x; y).$$

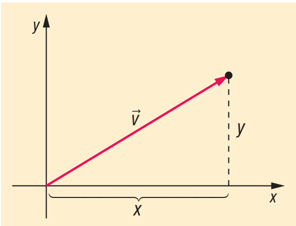


Figure 18

The length of a position vector $\vec{v}$ can be obtained easily if we apply the Pythagorean theorem for the right-angled triangle shown in the figure. ***(Figure 18)***

It implies that

$$|\vec{v}| = \sqrt{x^2 + y^2}.$$

Example 1

Let us calculate the length of the vector $\vec{a}(8; -6)$.

Solution I

If we represent the vector with the origin as the starting point in the coordinate system, then the coordinates of its end-point are $(8; -6)$. ***(Figure 19)***

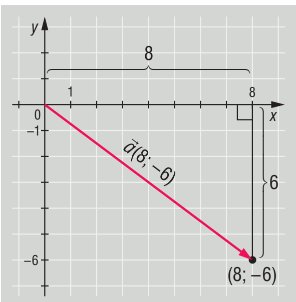


Figure 19

In the figure it can be seen that the length of $\vec{a}$ is the length of the hypotenuse of a right-angled triangle with 6- and 8-unit-long legs. Let us apply the Pythagorean theorem.

$$|\vec{a}| = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10.$$

Solution II

As we could see it earlier the dot product of a vector with itself – based on the definition – is equal to the square of the magnitude of the vector. Thus

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a} = 8^2 + (-6)^2 = 100, \quad \text{which implies} \quad |\vec{a}| = 10.$$

In general:

The length (magnitude) of the vector $\vec{a}(a_1; a_2)$ is:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}.$$

$|\mathbf{i}| = |\mathbf{j}| = 1$ and $\mathbf{i} \cdot \mathbf{j} = 0$

We know that base vectors are unit vectors and perpendicular to each other. It implies that their dot product is 0.

It is known that the operations of vectors can also be defined with the help of coordinates.

$$\vec{a} \pm \vec{b} = (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\mathbf{j}$$

$$\lambda \cdot \vec{a} = (\lambda \cdot a_1)\mathbf{i} + (\lambda \cdot a_2)\mathbf{j}$$

Let us examine the relationship between the coordinates and the dot product of the vectors.

Let us consider two vectors in the coordinate system according to the figure ***(Figure 20)***.

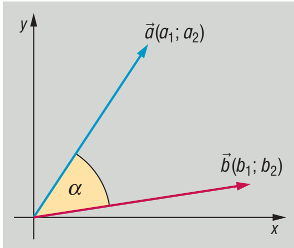


Figure 20

By applying the characteristics of the dot product operation:

$$\vec{a} \cdot \vec{b} = (a_1\mathbf{i} + a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) =$$
$$= a_1b_1\mathbf{i} \cdot \mathbf{i} + a_1b_2\mathbf{i} \cdot \mathbf{j} + a_2b_1\mathbf{j} \cdot \mathbf{i} + a_2b_2\mathbf{j} \cdot \mathbf{j}.$$

Let us use now that $\mathbf{i}^2 = \mathbf{j}^2 = 1$ and $\mathbf{i} \cdot \mathbf{j} = 0$, therefore the values of the two middle terms in the right-hand-side expression are 0, i.e. the dot product has the following simple form in terms of the coordinates:

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2.$$

Let us apply the definition of the dot product:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha = \sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2} \cdot \cos\alpha.$$

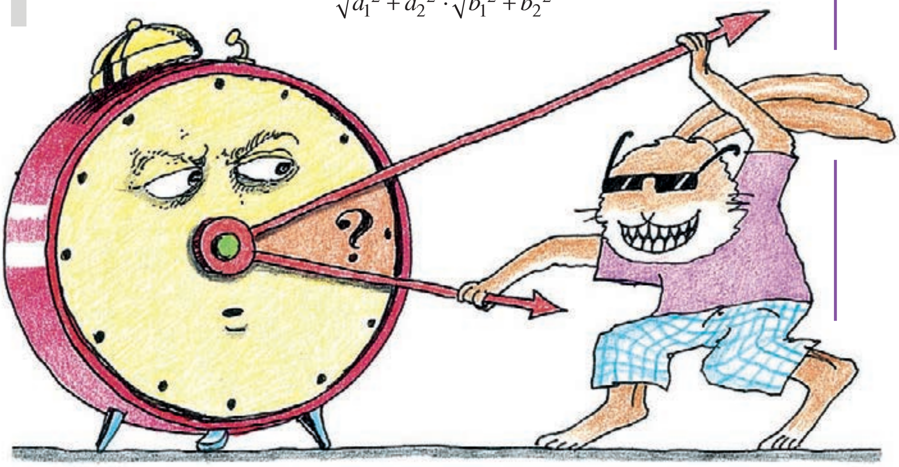
The result makes it possible to determine ***the angle of inclination of two vectors*** in the coordinate system since:

$$\sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2} \cdot \cos\alpha = a_1b_1 + a_2b_2.$$

It implies that it is true for the angle of inclination of two vectors that:

angle of inclination of two vectors

$$\cos\alpha = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2}}.$$



Example 2

Let us determine the value of x, if we know that the vectors $\vec{a}(3; 2)$ and $\vec{b}(x; -2)$ are perpendicular to each other.

Solution

Since the two vectors are perpendicular, their dot product is 0.

By expressing it with the help of the coordinates:

$$\vec{a} \cdot \vec{b} = 3x + 2 \cdot (-2) = 3x - 4 = 0.$$

Expressing x from the equation implies that $x = \frac{4}{3}$, i.e.: $\vec{b}\left(\frac{4}{3}; -2\right)$.

Example 3

Let us give the angle included between the vectors $\vec{a}(3; 0)$ and $\vec{b}(1; \sqrt{3})$.

Solution

Let us write down the dot product of the vectors with the help of the coordinates and then with the help of the definition:

$$\vec{a} \cdot \vec{b} = 3 \cdot 1 + 0 \cdot \sqrt{3} = 3;$$

$$\vec{a} \cdot \vec{b} = \sqrt{9} \cdot \sqrt{1+3} \cdot \cos\alpha = 6 \cdot \cos\alpha.$$

Since the two values need to be equal to each other:

$$6 \cdot \cos\alpha = 3, \quad \text{i.e.} \quad \cos\alpha = \frac{1}{2} \quad (0° \le \alpha \le 180°).$$

It implies that the measure of the angle included between the two vectors is: $\alpha = 60°$. ***(Figure 21)***

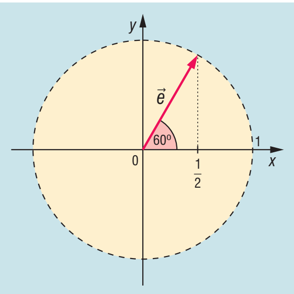


Figure 21

Example 4

Let us verify that the following inequality holds for arbitrary real numbers:

$$ab + cd \le \sqrt{a^2 + c^2} \cdot \sqrt{b^2 + d^2}.$$

Solution

Let us realise that the expressions of the inequality can also be found in the dot product of the vectors.

Let us consider the following two vectors: $\vec{u}(a; c)$ and $\vec{v}(b; d)$. Let us express the dot product of these both ways:

$$\vec{u} \cdot \vec{v} = ab + cd = \sqrt{a^2 + c^2} \cdot \sqrt{b^2 + d^2} \cdot \cos\alpha \le \sqrt{a^2 + c^2} \cdot \sqrt{b^2 + d^2}.$$

When writing down the inequality we considered the range of the cosine function, i.e. that $\cos\alpha \le 1$. It verifies our statement.

It is also worth examining under which conditions the equality holds. It is also true if $\cos\alpha = 1$, i.e. $\alpha = 0°$. For the vectors written down this condition means that they are unidirectional vectors, i.e. one of them can be derived from the other one by multiplying it with a suitable positive real number.

Let this real number be $\lambda \ne 0$.

Then $\vec{u}(a; c) = \lambda \cdot \vec{v}(b; d)$. Thus it is true for the corresponding coordinates that

$$a = \lambda \cdot b \quad \text{and} \quad c = \lambda \cdot d, \quad \text{which implies} \quad \frac{a}{b} = \frac{c}{d}$$

as the condition of the equality, if none of the denominators is 0.

It can easily be seen that in the case of $b = 0$ $a = 0$, and in the case of $d = 0$ $c = 0$ equality is obtained.

Notes:

1. Any vector of the space can unambiguously be expressed as the linear combination of the base vectors of a spatial coordinate system. ***(Figure 22)***
   For the base vectors **i**, **j** and **k** it is true that
   $$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1;$$
   $$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0.$$
2. The length (magnitude) of a vector $\vec{v}$ is:
   $$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}.$$
3. The dot product of two vectors is as follows if $\vec{v} = x_1 \cdot \mathbf{i} + y_1 \cdot \mathbf{j} + z_1 \cdot \mathbf{k}$, $\vec{u} = x_2 \cdot \mathbf{i} + y_2 \cdot \mathbf{j} + z_2 \cdot \mathbf{k}$:
   $$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\varphi,$$
   where φ is the angle included between the two vectors, or
   $$\vec{u} \cdot \vec{v} = x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2.$$
4. With the method applied in example 4 it can be verified that it is true for arbitrary real numbers $x_1, x_2, y_1, y_2, z_1, z_2$ that:
   $$\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2} \geq x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2.$$

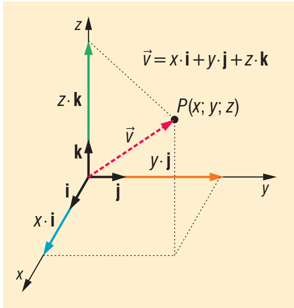


Figure 22

Exercises

1. The vectors $\vec{a}(3; 1)$ and $\vec{b}(2; -1)$ are given. Determine the results of the following operations:
   a) $\vec{a} \cdot \vec{b}$; *b)* $(\vec{a} + \vec{b}) \cdot \vec{a}$; *c)* $(\vec{a} - 2 \cdot \vec{b})^2$; *d)* $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$.
2. Determine the angle of inclination of the following two vectors:
   a) $\vec{a}(3; 1)$ and $\vec{b}(-1; 3)$; *b)* $\vec{a}(3; 1)$ and $\vec{b}(-3; -1)$;
   c) $\vec{a}(3; 1)$ and $\vec{b}(2; -1)$; *d)* $\vec{a}(3; 1)$ and $\vec{b}(-2; -1)$.
3. Give the value of the *x*-coordinate in the following vectors so that the given vectors will be perpendicular to each other.
   a) $\vec{a}(3; 1)$ and $\vec{b}(1; x)$; *b)* $\vec{a}(3; 1)$ and $\vec{b}(2x - 2; -1)$;
   c) $\vec{a}(3; 3x - 5)$ and $\vec{b}(2; -1)$; *d)* $\vec{a}(x^2 - x + 1; 1)$ and $\vec{b}(2; -1)$.
4. Give the coordinates of the unit vector unidirectional with the vector $\vec{a}(3; 4)$.
5. Two points are given in the coordinate system: $A(1; 5)$ and $B(6; 2)$. Give the angles of the triangle *OAB*, where *O* denotes the centre of the coordinate system.
6. Give the measure of the work done by the force $\vec{F}(3; 7)$, if there is a displacement of $\vec{s}(15; 6)$.
7. Verify the following inequalities. When is the equality satisfied?
   a) $4a + 3b \leq 5 \cdot \sqrt{a^2 + b^2}$; *b)* $4a + 3b \leq \sqrt{a^2 + 9} \cdot \sqrt{b^2 + 16}$.

Coordinate geometry

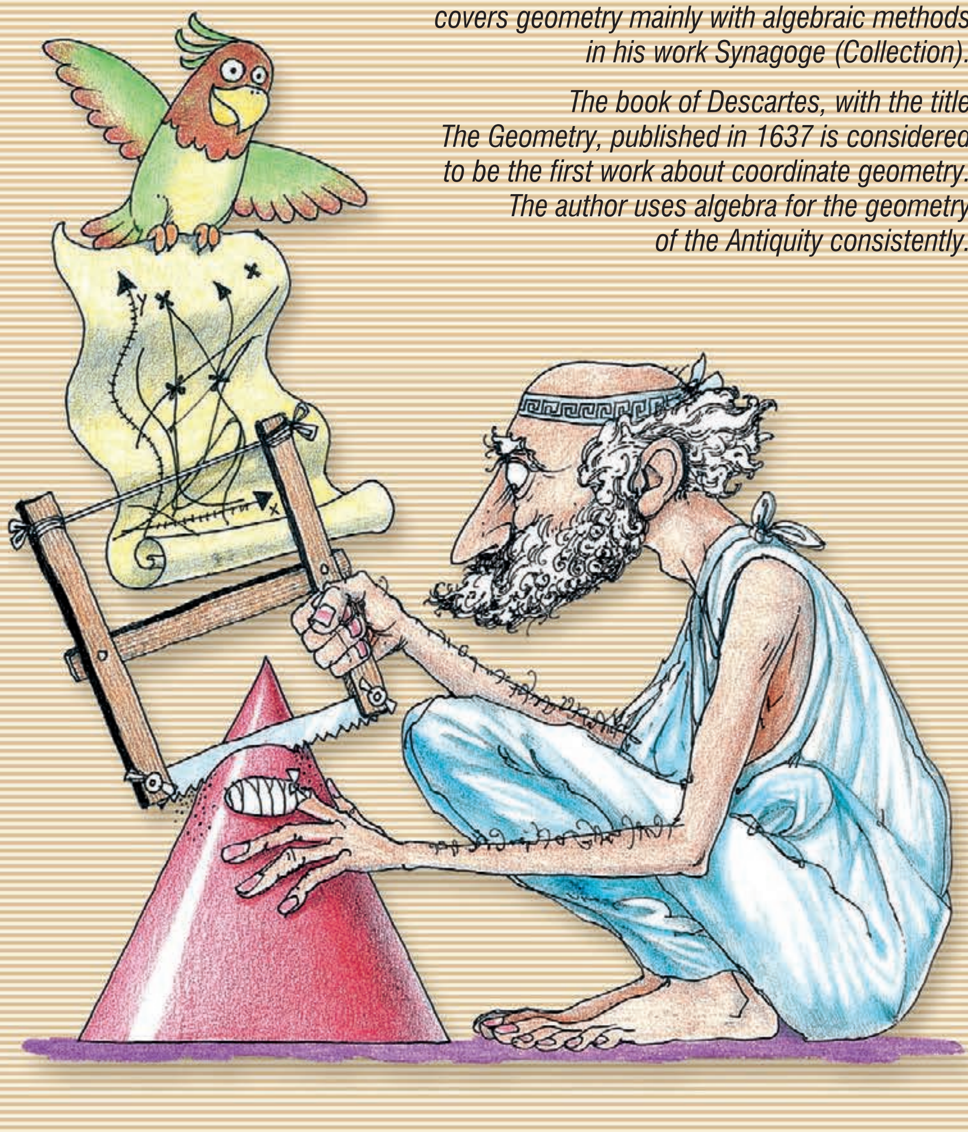
The fundamental feature of coordinate geometry is its ability to solve geometric problems and exercises with algebraic methods and with the help of a coordinate system.

This algebraic approach to geometry first appeared in the eight-volume book of Apollonius about conic sections in the 3rd century BC.

Hipparchus (2nd century BC) used the so called spherical coordinates to determine certain places on Earth.

Pappus of Alexandria (4th century AD) covers geometry mainly with algebraic methods in his work Synagoge (Collection).

The book of Descartes, with the title The Geometry, published in 1637 is considered to be the first work about coordinate geometry. The author uses algebra for the geometry of the Antiquity consistently.



1. Vectors in the coordinate system. Operations with vectors given with coordinates (reminder)

To discuss the basic relations and methods of coordinate geometry we will need what we have learnt so far about vectors in a coordinate system.

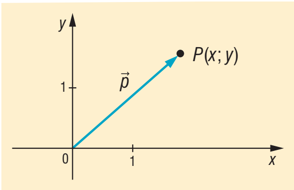


Figure 1

position vector

Position vector, base vectors, coordinates of a vector

DEFINITION: **In the Cartesian coordinate system the *position vector* of the point $P(x; y)$ is the vector pointing from the origin to the point.** ***(Figure 1)***

THEOREM: **If i is the position vector of the point (1; 0), and j is the position vector of the point (0; 1), then any vector $\vec{a}$ of the plane can unambiguously be expressed in the form $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j}$ (as the linear combination of the vectors i and j).** ***(Figure 2)***

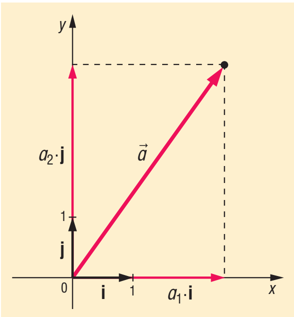


Figure 2

i and **j** are the **base vectors** of the coordinate system, and the real numbers a_1 and a_2 are the **coordinates** of the vector $\vec{a}$. Notation: $\vec{a}(a_1; a_2)$.

The coordinates of a vector in the coordinate system are the same as the coordinates of the end-point of its representative the starting point of which is the origin. It implies that in the coordinate system the coordinates of a given point and the coordinates of the position vector of this point are the same.

Operations with vectors given with coordinates

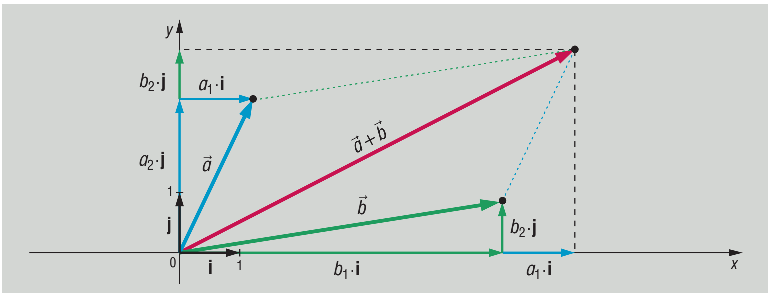
1. The coordinates of the sum of two vectors

If $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\vec{a} + \vec{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j},$$

i.e. the corresponding coordinates of the sum vector are obtained as the sum of the corresponding coordinates of the vectors to be added. ***(Figure 3)***

Figure 3



2. The coordinates of the difference of two vectors

If $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\vec{a} - \vec{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j},$$

i.e. the corresponding coordinates of the difference vector are obtained as the difference of the corresponding coordinates of the two vectors. ***(Figure 4)***

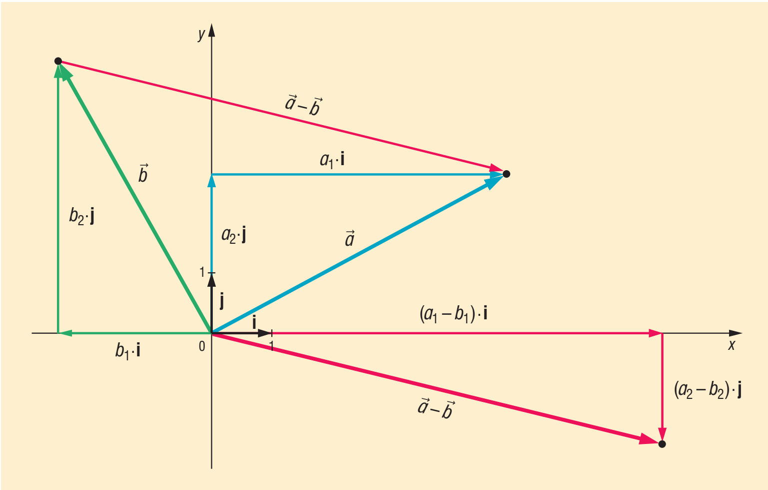


Figure 4

3. The coordinates of the scalar multiple of a vector

If $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and α is a real number, then

$$\alpha \cdot \vec{a} = (\alpha \cdot a_1)\mathbf{i} + (\alpha \cdot a_2)\mathbf{j},$$

i.e. the coordinates of α times the vector are α times the coordinates.

The identities below are valid for the multiplication of vectors by real numbers (scalar multiplication):

$$\alpha \cdot \vec{a} + \beta \cdot \vec{a} = (\alpha + \beta) \cdot \vec{a},$$

$$\alpha \cdot (\beta \cdot \vec{a}) = (\alpha \cdot \beta) \cdot \vec{a},$$

$$\alpha \cdot (\vec{a} + \vec{b}) = \alpha \cdot \vec{a} + \alpha \cdot \vec{b}.$$

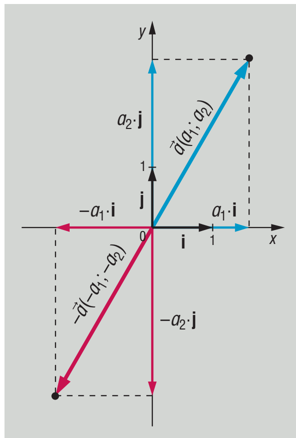


Figure 5

4. The coordinates of the negative of a vector

The vector opposite to the vector $\vec{a}(a_1; a_2)$ is the vector $-\vec{a}(-a_1; -a_2)$, i.e. the coordinates of this vector are –1 times the coordinates of the original vector. ***(Figure 5)***

5. The dot product of two vectors

dot product or scalar product

The *dot product* of the vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha,$$

where α is the measure of the angle included between the representatives of the vectors with a common starting point ($0° \le \alpha \le 180°$).

6. The dot product expressed in terms of the coordinates

If $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2,$$

i.e. the dot product of the two vectors is the sum of the product of the corresponding coordinates.

Exercises

1. Draw a coordinate system and draw the position vectors of the following points.

a) $A(1; 1)$; *b)* $B(-1; 2)$; *c)* $C(-3; -4)$;
d) $D(0; -6)$; *e)* $E(-2; 4)$; *f)* $F(3; -5)$.

2. Give the position vectors of the points given in the previous exercise in terms of the base vectors **i** and **j** (as the linear combination of the vectors **i** and **j**).

3. The vectors $\vec{a}(1; 2)$, $\vec{b}(0; 5)$, $\vec{c}(-2; 3)$, $\vec{d}(7; -9)$ are given. Give the coordinates of the following vectors.

a) $\vec{a} + \vec{b}$; *b)* $\vec{b} + \vec{c} + \vec{d}$; *c)* $\vec{d} - \vec{b}$;
d) $\vec{a} + \vec{b} - \vec{c}$; *e)* $2 \cdot \vec{c}$; *f)* $-3 \cdot \vec{d}$;
g) $\frac{1}{3} \cdot \vec{a}$; *h)* $-\frac{3}{5} \cdot \vec{d}$; *i)* $3 \cdot \vec{a} + 2 \cdot \vec{b}$;
j) $3 \cdot \vec{c} - 5 \cdot \vec{d}$; *k)* $2 \cdot (-2 \cdot \vec{a} + 6 \cdot \vec{d})$; *l)* $-\frac{3}{4} \cdot (7 \cdot \vec{c} - 5 \cdot \vec{a})$.

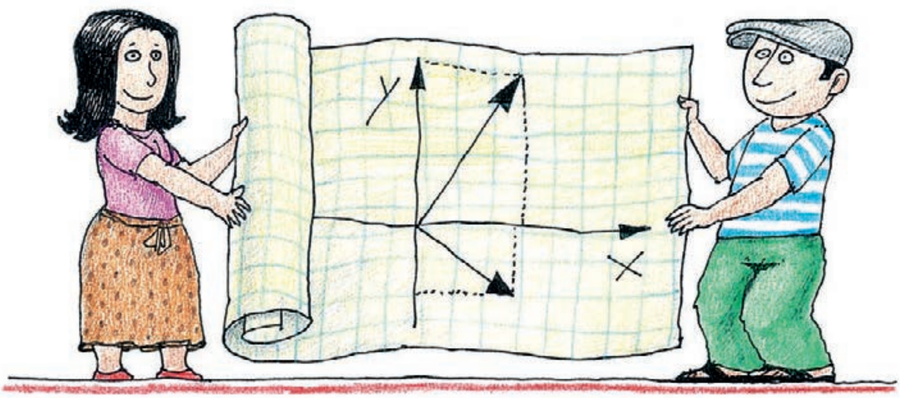
4. Calculate the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$, if

a) $A(0; 1)$, $B(3; 2)$; *b)* $A(4; 1)$, $B(-1; 6)$;
c) $A(-2; -5)$, $B(7; -10)$; *d)* $A(8; -7)$, $B(-4; 5)$.

5. Give the dot product of the vectors $\vec{a}$ and $\vec{b}$, if

a) $\vec{a}(1; 2)$, $\vec{b}(4; 1)$; *b)* $\vec{a}(5; -2)$, $\vec{b}(3; 0)$;
c) $\vec{a}(-7; 5)$, $\vec{b}(4; -9)$; *d)* $\vec{a}(-6; -10)$, $\vec{b}(3; -11)$.

6. Calculate the value of x, if we know that the vectors $\vec{a}(5; 7)$ and $\vec{b}(4; -x)$ are perpendicular to each other.



11. The parabola and the quadratic function (higher level courseware)

In year 9 when having dealt with the quadratic functions we said that the graph of any quadratic function defined on the set of real numbers is a parabola, and we did prove it for the graph of the function $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Using the coordinate geometry knowledge we have just gained we can also prove the general case.

THEOREM: **The graph of any quadratic function defined on the set of real numbers is a parabola the axis of which is parallel with the *y*-axis.**

Proof

Let us consider the quadratic function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$, and let us assume that $a > 0$. The equation of the graph of f is $y = ax^2 + bx + c$. We transform the right-hand-side of this equation by completing the square:

$$y = ax^2 + bx + c = a \cdot \left(x^2 + \frac{b}{a} \cdot x + \frac{c}{a}\right) = a \cdot \left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] =$$

$$= a \cdot \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

This implies that

$$y + \frac{b^2 - 4ac}{4a} = a \cdot \left(x + \frac{b}{2a}\right)^2.$$

By comparing it with the equation

$$y - v = \frac{1}{2p} \cdot (x - u)^2$$

of the parabola "open upwards" with the vertex $T(u; v)$, with the focal parameter p, we obtain that

$$u = -\frac{b}{2a}, \quad v = -\frac{b^2 - 4ac}{4a}, \quad p = \frac{1}{2a}, \quad \text{i.e.}$$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$ $(a > 0)$ is a parabola with the axis parallel with the *y*-axis, with the vertex $T\left(-\frac{b}{2a}; -\frac{b^2 - 4ac}{4a}\right)$ and with the focal parameter $\frac{1}{2a}$.

The proof is similar for the case $a < 0$. Our statements relating to the parabola change only in the way that in this case the focal parameter of the parabola is $-\frac{1}{2a}$.

With this we have proven the theorem.

The converse of theorem just proven is also true:

THEOREM: **There is exactly one quadratic function defined on the set of real numbers for any parabola with the axis parallel with the *y*-axis the graph of which is the parabola considered.**

Proof

Let us consider a parabola “open upwards” with the axis parallel with the y-axis, with the vertex $T(u;\, v)$ and with the focal parameter p. Its equation is

$$y - v = \frac{1}{2p} \cdot (x - u)^2.$$

Let us rearrange this equation so that the left-hand-side is equal to y, then let us rearrange the right-hand-side:

$$y = \frac{1}{2p} \cdot (x - u)^2 + v = \frac{1}{2p} \cdot (x^2 - 2ux + u^2) + v =$$

$$= \frac{1}{2p} \cdot x^2 - \frac{u}{p} \cdot x + \frac{u^2 + 2pv}{2p}.$$

The equation obtained is the equation of the graph of the quadratic function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2p} \cdot x^2 - \frac{u}{p} \cdot x + \frac{u^2 + 2pv}{2p}$. In the case of a parabola “open downwards” the initial equation is

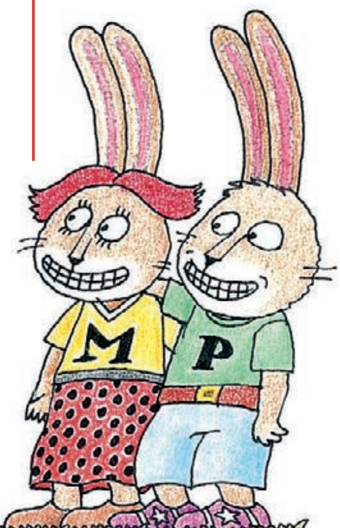
$$y - v = -\frac{1}{2p} \cdot (x - u)^2$$

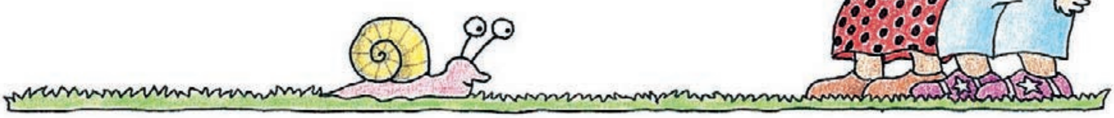
the proof is done in a completely analogous.

We are summarising the two theorems proven above with a slightly different formulation.

Let $\boldsymbol{M}$ denote the set of quadratic functions defined on the set of real numbers, and let $\boldsymbol{P}$ denote the set of parabolas with the axis parallel with the y-axis.

THEOREM: **One can give a one-to-one assignment between the elements of the sets *M* and *P*, i.e. there is one and only one parabola from the set *P* for every function from the set *M*, which is the graph of the function considered, and vice versa, there is one and only one function from the set *M* for every parabola from the set *P* the graph of which is the parabola considered.**





Exercises

1. The two zeros of the quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ are

a) $0; 4$; *b)* $-2; 3$; *c)* $-4; 9$;

d) $-\sqrt{2}; \sqrt{5}$; *e)* $-\frac{3}{4}; \frac{5}{8}$.

Give the equation of the axis of symmetry of the graph.

2. The quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ takes its minimum at the place x_0. Give the coordinates of the vertex of the graph, if

a) $x_0 = 1,\ f(x_0) = -2$; *b)* $x_0 = -3,\ f(x_0) = 5$;

c) $x_0 = 8,\ f(x_0) = -5$.

3. Plot the graphs of the following quadratic functions defined on the set of real numbers in the Cartesian coordinate system.

a) $x \mapsto x^2$; *b)* $x \mapsto \left(\frac{x}{2}\right)^2 - 3$; *c)* $x \mapsto -2(x-3)^2$; *d)* $x \mapsto 2x^2 + 2x - 12$.

Give the focal parameter, the coordinates of the vertex and the focus, and the equation of the directrix of the parabola obtained in each case.

4. For which quadratic function defined on the set of real numbers is it true that the graph of it is the parabola with the following equation:

a) $6y - 12 = -(x+1)^2$; *b)* $y + 3 = -2\cdot(x-1)^2 + 5$; *c)* $8y - 2\cdot(x-5)^2 = 12$?

5. Set up the equation of the parabola with the axis parallel to the y-axis three points of which are $A(-2; 0)$, $B(4; 0)$, $C(1; -5)$. . Give the quadratic function the graph of which is this parabola.