colourful

Mathematics

9

MOZAIK

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Mathematics

textbook

9

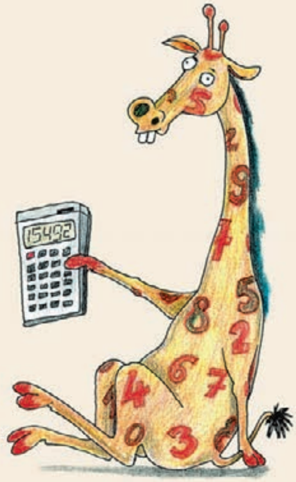
Mozaik Education – Szeged, 2015

Combinatorics, sets

Algebra and arithmetics

Functions

Triangles, quadrilaterals, polygons



Equations, inequalities, simultaneous equations

Congruent transformations

Statistics

Guide to use the course book

The notations and highlights used in the book help with acquiring the courseware.

- The train of thought of the worked examples show samples how to understand the methods and processes and how to solve the subsequent exercises.
- The most important definitions and theorems are denoted by colourful highlights.
- The parts of the courseware in small print and the worked examples noted in claret colour help with deeper understanding of the courseware. These pieces of knowledge are necessary for the higher level of graduation.
- Figures, the key points of the given lesson, review and explanatory parts along with interesting facts of the history of mathematics can be found on the margin.

The difficulty level of the examples and the appointed exercises is denoted by three different colours:

Yellow: drilling exercises with basic level difficulty; the solution and drilling of these exercises is essential for the progress.
Blue: exercises the difficulty of which corresponds to the intermediate level of graduation.
Claret: problems and exercises that help with preparing for the higher level of graduation.

These colour codes correspond to the notations used in the Colourful mathematics workbooks of Mozaik Education. The workbook series contains more than 3000 exercises which are suitable for drilling, working on in lessons and which help with preparing for the graduation.

The end results of the appointed exercises can be found on the following website: *www.mozaik.info.hu*. Website *www.mozaweb.hu* offers more help material for processing with the course book.

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And here are a few pieces of advice to follow and some notes from those who were experts of and acquired deep knowledge of the mathematical science. Let these give the basics and serve as encouragement for those who would like to acquire the methods and relations mentioned in the book.

"Problem-solving is a matter of practice just as swimming, skiing or playing the piano. You can acquire it only by copying and practising. I cannot give you a key which opens all doors and solves all problems, but I can give good examples which can be copied and also give several occasions for practising. Who wants to learn swimming has to jump into the water, who wants to learn how to solve problems has to practice problem-solving." (György Pólya)

"What we have to find out for ourselves leaves a mark in our memory; we can follow this mark again once we need to." (G.C. Lichtenberg)

"In mathematics – unlike in other fields of life according to our day-to-day experience – generally who deserves it has luck." (Alfréd Rényi)

"Mathematics is a game where – unlike in chess – we can move one step back. Only the last step counts." (Pál Erdős)

"It is an experimental fact that the world is moving according to the laws of logical thinking. … But I do not know why the world is created so that one can find his/her way around in it by thinking logically." (R. L. Dobrusin)

The Authors wish productive work and learning.

Combinatorics, sets

Combinatorics, the "science of combination" usually deals with counting objects. It was Leibniz who first systematized the combinatorial knowledge, and then Jacob Bernoulli applied it when solving probabilistic problems. In the middle of the twentieth century this branch of mathematics advanced considerably due to the practical applicability.

5. Order of sets, inclusion-exclusion principle

The notation of the order (or cardinal number) of set A: $|A|$.

For example:
A = {two-digit square numbers}, $|A| = 6$;
B = {the chess figures on the board at the start of the game}, $|B| = 32$.

Example 1

In a class of 30 fifteen pupils learn how to play the piano, six pupils learn how to play the violin and two pupils learn how to play both the piano and the violin. How many pupils of the class do not learn how to play either the piano or the violin?

Solution I

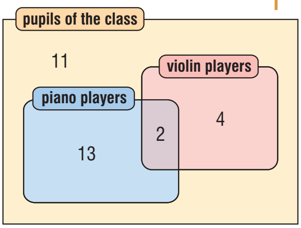


Figure 30

Let us draw a Venn diagram and put the number of pupils belonging to each group into the corresponding parts. *(Figure 30)*

We put 2 into the intersection of the piano and violin players. As out of the 15 piano players 2 also play the violin, $15 - 2 = 13$ pupils play the piano but do not play the violin. Out of the 6 violin players 2 also play the piano, so $6 - 2 = 4$ pupils play the violin but do not play the piano. The number of those pupils who do not play the piano and do not play the violin either is: $30 - [(15 - 2) + (6 - 2) + 2] = 30 - 19 = 11$.

Solution II

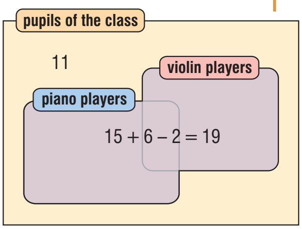


Figure 31

If we add the number of piano players and violin players, we counted twice those pupils who learn how to play both instruments, so we have to take away their number once. Thus the number of those pupils who play the piano or play the violin is: $15 + 6 - 2 = 19$, therefore $30 - 19 = 11$ pupils of the class do not play either the piano or the violin.

Let us introduce the following notations: let the set of the pupils of the class be: U; the set of the piano players be: A; the set of the violin players be: B. *(Figure 31)*.

Using these notations the above solution is as follows:

$$|A \cup B| = |A| + |B| - |A \cap B| \quad \text{and} \quad |\overline{A \cup B}| = |U| - |A \cup B|. \qquad (1)$$

Example 2

In the course of a survey 100 people are being asked what source they get the news from. The following results were obtained:

TV: 65; radio: 38; newspaper: 39; TV and radio: 27; TV and newspaper: 20; radio and newspaper: 9; TV, radio and newspaper: 6.

How many of the people being asked do not get the news from any of the mentioned sources? How many of the people being asked get the news from only one source of the mentioned three sources?

Solution I

Let us draw a Venn diagram and put the number of people belonging to each group into the corresponding parts. ***(Figure 32)***

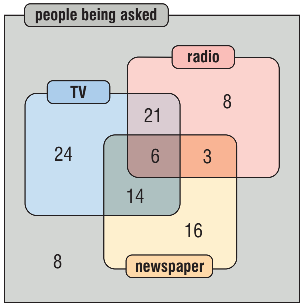


Figure 32

- **Step 1:** Firstly we put 6 into the intersection of the three sets.
- **Step 2:** Then we put in those who get the news from the TV and the radio but not from the newspapers; their number is 27 – 6 = 21. Then we put in those who get the news from the TV and the newspapers but not from the radio; their number is 20 – 6 = 14. Then we put in those who get the news from the radio and the newspapers but not from the TV; their number is 9 – 6 = 3.
- **Step 3:** Now we can calculate the number of people who get the news
  only from the TV: 65 – 21 – 6 – 14 = 24,
  only from the newspapers: 39 – 14 – 6 – 3 = 16,
  only from the radio: 38 – 21 – 6 – 3 = 8.

Therefore 24 + 16 + 8 = 48 of the people being asked get the news from only one source.

- **Step 4:** 24 + 8 + 16 + 14 + + 21 + 3 + 6 = 92 people being asked get the news from at least one of the three sources, thus the number of people who do not get the news from any of the mentioned sources is: 100 – 92 = 8.

Solution II

Let U denote the set of people being asked, let A denote the set of people who get the news from the TV, B the set of those who get the news from the radio, C the set of those who get the news from the newspapers. ***(Figure 33)***

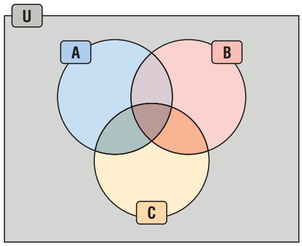


Figure 33

Firstly we calculate the number of people who get the news from at least one of the three sources: If we add the number of people getting the news from the TV, from the radio and from the newspapers (65 + 38 + 39), then we counted twice those who get the news from both the TV and the radio, from both the TV and the newspapers, from both the radio and the newspapers, so their number should be taken away: 65 + 38 + 39 – 27 – 20 – 9. But the number of those who get the news from all three sources was added three times, then it was subtracted three times, thus we have to add it once; so the number of people who get the news from at least one of the three sources is: 65 + 38 + 39 – 27 – 20 – 9 + 6 = 92.

This train of thought using set notation is as follows:

$$|A \cup B \cup C| = |A| + |B| + |C| -$$
$$- |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \qquad (2)$$

People who do not get the news from any of the mentioned sources: 100 – 92 = 8, by using sets:

$$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| =$$
$$= |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|.$$

Now we look for the number of people who get the news only from the TV. All together 65 people get the news from the TV, but the number of people who get the news from both the TV and the newspapers and the number of people who get the news from both the TV and the radio should be taken away: 65 – 20 – 27. In this case we subtracted the number of people who get the news from all three sources twice, thus their number should be added once; so the number of people who get the news only from the TV is 65 – 20 – 27 + 6 = 24.

Written using the notation of sets:

$$|A \cap \overline{B} \cap \overline{C}| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$

Similarly the number of people who get the news only from the radio is: 38 – 27 – 9 + 6 = 8,

$$|\overline{A} \cap B \cap \overline{C}| = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|.$$

And the number of people who get the news only from the newspapers is: 39 – 20 – 9 + 6 = 16,

$$|\overline{A} \cap \overline{B} \cap C| = |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Therefore 24 + 16 + 8 = 48 of the people being asked get the news from only one source.

inclusion-exclusion principle

The relations denoted by equations (1) and (2) are called the ***inclusion-exclusion principle.***

$$|A \cup B| = |A| + |B| - |A \cap B| \quad \text{and} \quad |\overline{A \cup B}| = |U| - |A \cup B| \qquad (1)$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \qquad (2)$$

Exercises

1. A pizza seller jotted down 100 orders in a row. 60 customers wanted both cheese and pepperoni, 80 customers wanted cheese and 72 customers wanted pepperoni on their pizza.
   a) How many people ordered pizza with cheese without pepperoni?
   b) How many people ordered pizza with pepperoni without cheese?
   c) How many people did not want either cheese or pepperoni on their pizza?
2. In one round of the basketball championship it was counted how many players scored points by 2 point field goals, by 3 point field goals and by penalty throw. 70 players threw 2 point field goals, 44 players threw 3 point field goals and 32 players scored points by penalty throw. 19 players threw both 2 point and 3 point field goals, 16 players threw 2 point field goals and scored points by penalty throw too. 21 players threw 3 point field goals and scored points by penalty throw, and 6 players scored points in all three ways.
   a) How many players threw only 2 point field goals?
   b) How many players scored points by 2 or 3 point field goals, but did not score points by penalty throws?
   c) How many players scored points by 2 or 3 point field goals?
   d) How many players did not score points by penalty throws?

3. 102 stairs lead up to a tower. Dolly goes up 1, Gladys 2 and Sue 3 stairs by one step. How many stairs are there on which exactly two of them step before they reach the top?

4. *a)* How many positive integers not greater than 100 are there which are not divisible by either 2 or 3?
   b) How many positive integers not greater than 100 are there which are not divisible by either 2 or 3 or 5?
   c) How many positive integers not greater than 100 are there which are not divisible by either 2 or 3 or 5 or 7?
   d) Count how many prime numbers there are which are not greater than 100.

5. At the school 80% of the pupils who attend the Maths club play basketball, and 30% of the basketball players attend the Maths club. If there are a total of 15 pupils in the Maths club, then how many pupils play basketball?

6. 100 pupils took part in a Mathematical Olympiad and there were 4 exercises to solve. The first problem was solved by exactly 90 pupils, the second by exactly 80, the third by exactly 70 and the fourth by exactly 60. None of the participants solved all four problems. Those who could solve the third and the fourth problem won a prize. How many pupils won a prize?

7. Out of the 20 pupils in our class 14 have brown eyes, 15 have dark hair, 17 are heavier than 50 kg, and 18 are taller than 160 cm. Show that at least 4 pupils bear all four characteristics.

8. The class has a new teacher. One of the pupils introduced the class in the following way: There are 45 pupils in the class, 25 of them are boys. The number of the excellent pupils is 30, 16 of them are boys. 28 pupils do sport, 18 of them are boys and 17 of them are excellent students. There are 15 boys who are excellent pupils and also do sport. The teacher told the reporting pupil that there was an error in the report. How could the teacher know it if the teacher had not met anyone from the class before?

9. In a company of 1000 married couples $\frac{2}{3}$ of the husbands taller than their wives are also heavier than their wives. $\frac{3}{4}$ of the husbands heavier than their wives are taller than their wives. If there are 120 wives who are taller and also heavier than their husbands, then how many husbands are taller and heavier than their wives?

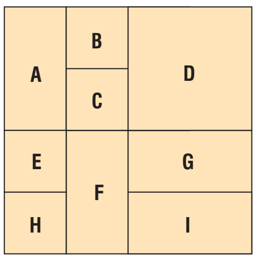
10. Out of four sets any two has an element in common; the intersection of any three sets is the empty set. Give sets which correspond to the conditions. Give sets with an order of three which correspond to the conditions. Is it possible to give five sets which correspond to the conditions? Is it possible to give five sets with an order of three which correspond to the conditions?

11. Give three subsets of the set of natural numbers for which it is true that the intersection of any two sets has infinitely many elements but the intersection of the three sets is empty.

Puzzle

We cover a square with congruent square-shaped paper sheets, the result can be seen in the figure.

In which order did we put the paper sheets down?

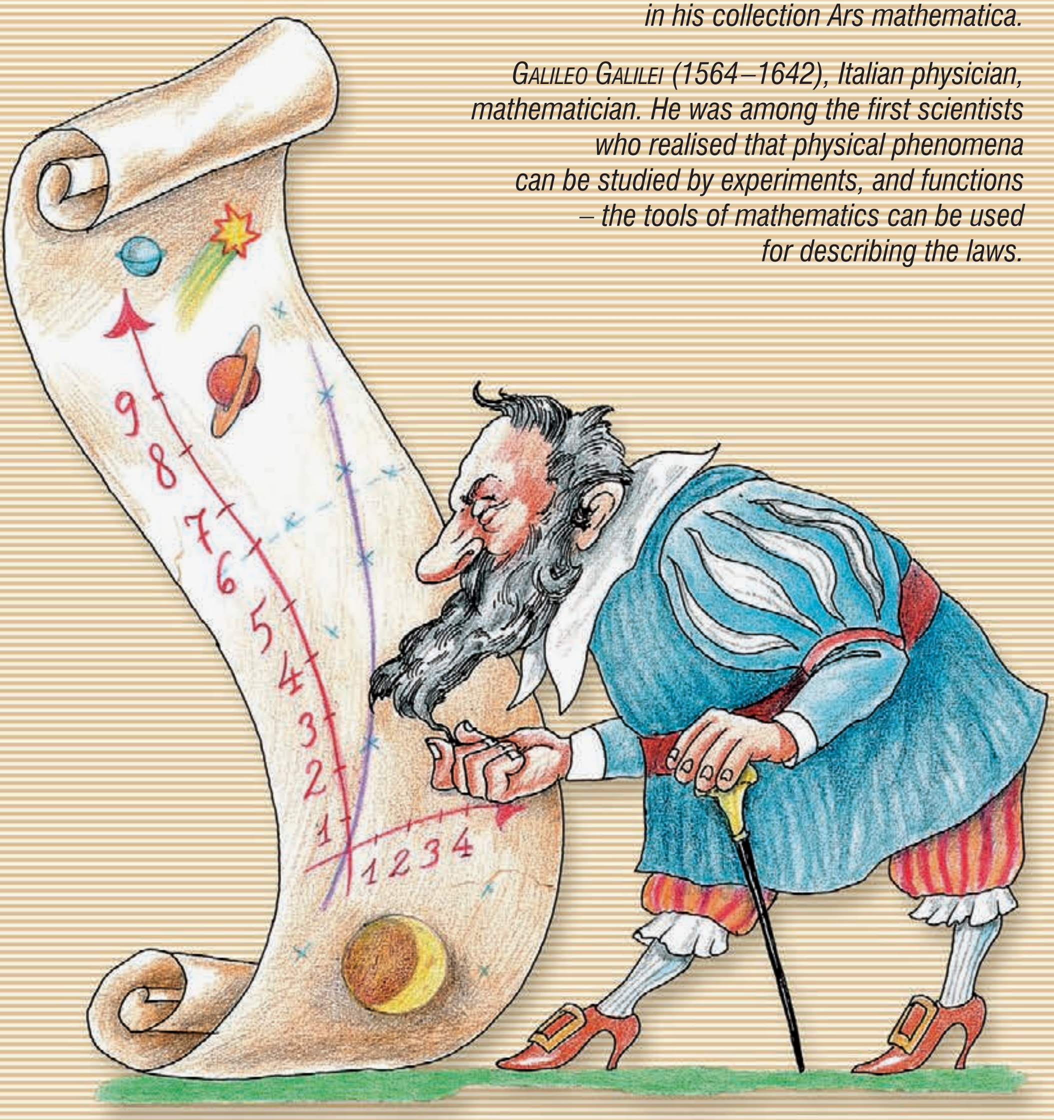


Functions

"...the fundamental laws of nature cannot be expressed in any other way than in mathematical form, in the form of relations between physical amounts which can be described by numbers. In other words: one can read the great book of Nature only if one knows the language this book was written in, and it is the mathematical language."

The above words were cited as the opinion of Galilei by Alfréd Rényi in the dialog "The language of the book of Nature" in his collection Ars mathematica.

Galileo Galilei (1564–1642), Italian physician, mathematician. He was among the first scientists who realised that physical phenomena can be studied by experiments, and functions – the tools of mathematics can be used for describing the laws.



4. The quadratic function

Henceforth we deal with functions where the largest appearing index of the variable x is the second index. These functions are called quadratic functions.

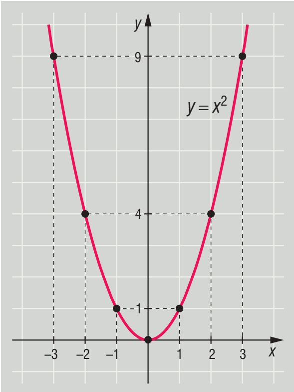


Figure 33

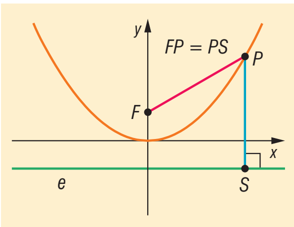


Figure 34

Example 1

The area of a square with side a (e.g. cm) is a^2 (cm^2). In connection with this we can interpret function $g(a) = a^2$ on the positive numbers. Let us plot the graph of function f which is an extension of this function to all real numbers, i.e. the following function:

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$$

Solution

The image of function f is a nice curve which is called ***parabola*** ***(Figure 33)***. In geometry later on we are going to learn that a parabola is the set of points in the plane which are equidistant from a given straight line e and from a given point F which does not lie on straight line e ***(Figure 34)***. Point F is the ***focus*** of the parabola; straight line e is the ***directrix*** of the parabola. The ***axis*** of the parabola received as the graph of function f is the y-axis, its ***vertex*** is the origin.

Let us characterise the function $f\colon \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

It is decreasing on $]-\infty; 0]$, it is increasing on $[0; +\infty[$, it has a minimum at $x = 0$, its value is $f(0) = 0$. It can be proved that the range of the function is the interval $[0; +\infty[$, i.e. the set of non-negative real numbers.

The graph of the function, the parabola is symmetric about the y-axis. The reason is simple, since $(-x)^2 = x^2$, so if a point $P(x; y)$ lies on the parabola, then the image $P'(-x; y)$ of point P when reflected about the y-axis also lies on the curve. This property of function f can also be formulated as follows: $f(-x) = f(x)$.

If $f(-x) = f(x)$, then
f is an even function

DEFINITION: The functions, where $f(-x) = f(x)$ is fulfilled for every x of the domain, are called *even functions*.

For example $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = |x|$ is an even function, since it is true that $g(-x) = g(x)$ is fulfilled for every element of the domain. The graph of function g is also symmetric about the y-axis.

Example 2

Let us plot the graph of the following $\mathbb{R} \to \mathbb{R}$ type functions:

$$g(x) = (x-2)^2; \qquad h(x) = x^2 - 4.$$

Solution

If we create the table of values to plot the graph of function g, we shortly realise that $(x-2)^2$ takes the same values at a place 2 greater than x^2 does.

For example it takes 0 at 2, 1 at 3, and 4 at 2. When plotting the graph it means that by translating the image of x^2 by 2 into the positive direction along the x-axis we get the image of $(x-2)^2$.

It is even more obvious from the definition of function h that we get the graph of x^2-4 from the graph of x^2 by translating it by 4 units into the negative direction along the y-axis. ***(Figure 35)***

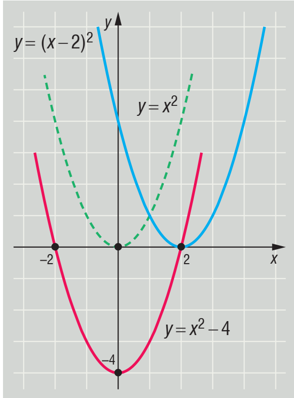


Figure 35

Example 3

Let us plot the graphs of and characterise the following $\mathbb{R} \to \mathbb{R}$ type functions:

$$f(x) = x^2 - 2x + 3; \qquad g(x) = 1 - x^2; \qquad h(x) = -x^2 + 4x - 3.$$

Solution

Function f can also be formulated as follows, i.e. by completing the square:

$$f(x) = (x^2 - 2x + 1) + 2 = (x-1)^2 + 2.$$

So the parabola, which is the image of function x^2, should be translated by 1 to the right and by 2 upwards (into the positive direction). ***(Figure 36)***

So function f is decreasing on the interval $]-\infty; 1]$, it is increasing on $[1; +\infty[$; it has a minimum at the place $x = 1$. Its minimum value is $f(1) = 2$. The range of f is interval $[2; +\infty[$.

Let us rewrite function g as follows: $g(x) = -x^2 + 1$, and let us first plot the graph of function $x \mapsto -x^2$. It differs from function x^2 in a way that its value is -1 times the value of the previous at all places, which means that the image of x^2 should be reflected about the x-axis. The image of g is derived from this by translating it by 1 unit upwards. ***(Figure 37)***

It is practical to rewrite the function h as follows:

$$h(x) = -(x^2 - 4x + 3) = -(x-2)^2 + 1.$$

When plotting the image of h using this form we get the graph shown in figure 38. Function h is increasing on the interval $]-\infty; 2]$, it is decreasing on the interval $[2; +\infty[$; it has a maximum at 2. Its maximum value is $h(2) = 1$. The range of h is interval $]-\infty; 1]$.

Figure 36

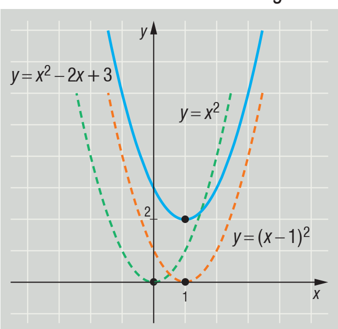


Figure 37

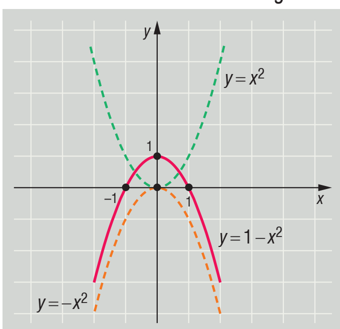
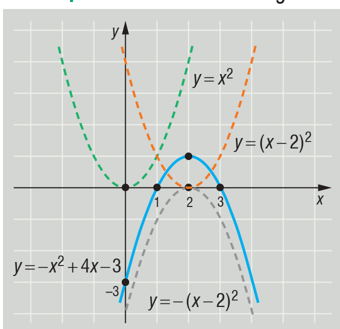


Figure 38



Example 4

Using the knowledge obtained about the functions let us solve the following exercise. Two ships are heading along two perpendicular straight lines towards the intersection point of the straight lines in the sea. At a given time point the first ship, which is cruising at a speed of 30 km/h, is 100 km away from intersection point *O* of the two straight lines. At the same time the second ship, which is cruising at a speed of 40 km/h, is 300 km away from point *O*. At what time will the two ships be the closest to each other and what is this shortest distance?

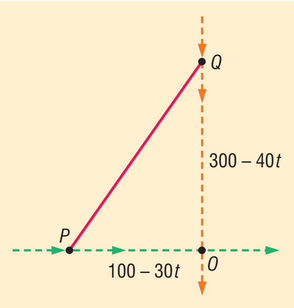


Figure 39

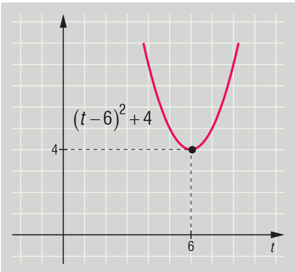


Figure 40

Solution

Let us denote the time measured in hours and elapsed since the given time point by t, and let us calculate the distance of the two ships at time point t. The first ship, the place of which at time point t is denoted by P, has covered $30t$ kilometres, thus its distance measured from O is $PO = 100 - 30t$. If point P was to the right of point O, then this distance would be $30t - 100$, thus the formula for the distance which is correct in both cases: $PO = |100 - 30t|$. ***(Figure 39)***

The distance of the second ship denoted by Q from point O is similarly $QO = |300 - 40t|$. From the right-angled triangle POQ using the Pythagorean theorem the square of distance PQ is:

$$PQ^2 = (100 - 30t)^2 + (300 - 40t)^2.$$

Obviously the square of the distance is the shortest exactly when the distance is the shortest. So the exercise can be formulated as follows: the function

$$f(t) = (100 - 30t)^2 + (300 - 40t)^2, \quad t \geq 0$$

is given, let us find its minimum place and its minimum value.

Let us first transform the expression defining the function:

$$(100 - 30t)^2 + (300 - 40t)^2 =$$
$$= 10\,000 - 6000t + 900t^2 + 90\,000 - 24\,000t + 1600t^2 = 2500(t^2 - 12t + 40).$$

So it is enough to find the minimum place of function

$$g(t) = t^2 - 12t + 40 = (t - 6)^2 + 4, \quad t \geq 0,$$

the value of f will be the smallest at the same place.

Let us plot the graph of function g ***(Figure 40)***. The image of function g is a parabola, which we get from the so called normal parabola (from the image of function x^2) by translating it by 6 to the right and by 4 upwards. The minimum place of g is at $t = 6$, its minimum value is $g(6) = 4$. According to this the minimum place of f is also at $t = 6$, its minimum value is $f(6) = 2500 \cdot 4 = 100^2$.

So the answer to the question of the exercise is as follows: the two ships will be closest to each other in 6 hours measured from the given time point, their distance will be 100 km.

We have already drawn the graph of function x^2, the (normal) parabola several times. We always draw it in a way that if we take the chord connecting two points of the parabola, then the curve between the two points will be below the chord ***(Figure 41)***. Is it really so?

We are going to show now that if $a < b$, and if we draw the chord connecting points $(a; a^2)$, $(b; b^2)$, then the point of the parabola at the mind-point of interval $[a; b]$ will indeed be below the chord. This property of the curve of function x^2 is expressed as the curve (from below) is **convex.** ***(Figure 42)***

Using the notations of the figure it is enough to show that $FR < FS$.

The length of line segment FR is: $\left(\frac{a+b}{2}\right)^2$.

Line segment FS is the mid-line of trapezium $ABQP$, thus $F = \frac{PA+QB}{2}$.

Since $PA = a^2$, $QB = b^2$, it is enough to show that $\left(\frac{a+b}{2}\right)^2 < \frac{a^2+b^2}{2}$.

Let us do the squaring on the left side, and let us multiply both sides of the inequality by 4:

$$a^2 + 2ab + b^2 < 2a^2 + 2b^2.$$

The right side is greater than the left side if and only if the difference of the right side and the left side is positive. And it is true, because the difference is:

$$a^2 - 2ab + b^2 = (b - a)^2 > 0,$$

since $b > a$. Since the steps can be reversed, the original statement is also true.

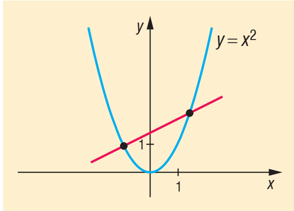


Figure 41

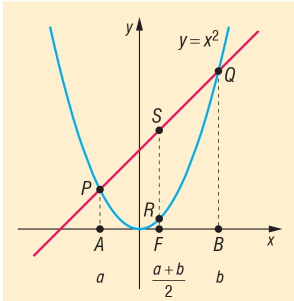


Figure 42

Exercises

1. Let us plot the graphs of and characterise the following $\mathbb{R} \to \mathbb{R}$ type functions.
   a) $f(x) = x^2 + 1$;
   b) $g(x) = -x^2$;
   c) $h(x) = (x - 1)^2$;
   d) $k(x) = -(x + 1)^2$;
   e) $l(x) = -x^2 + 4$.

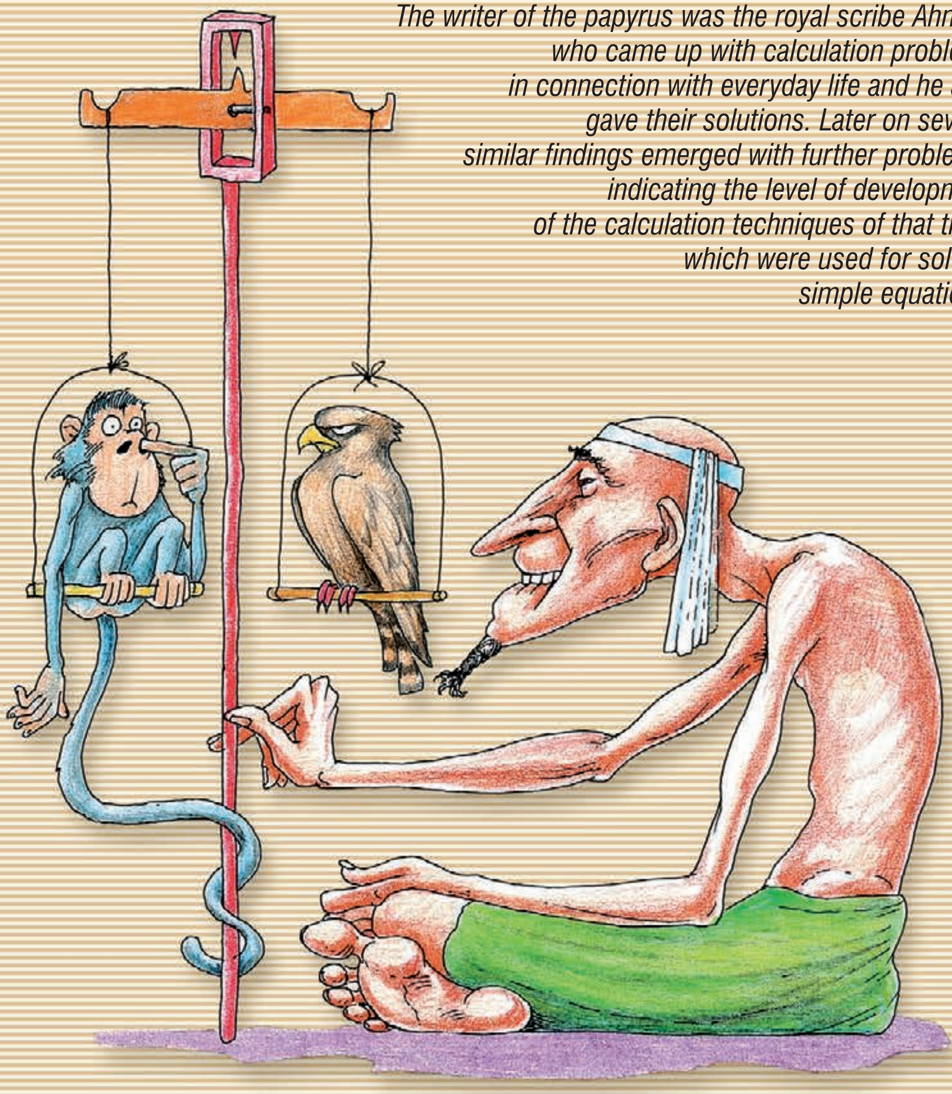
2. Let us plot the graphs of and characterise the following $\mathbb{R} \to \mathbb{R}$ type functions.
   a) $f(x) = 2x^2$;
   b) $g(x) = \frac{1}{2}x^2$;
   c) $h(x) = x^2 - 6x + 5$;
   d) $k(x) = -x^2 - 4x + 2$.

3. We throw a stone vertically upward with a velocity of 20 m/s. What distance does it go upward and after what time does it fall back to the ground? (Let us choose 10 m/s^2 rounded as the acceleration due to gravity, and let us neglect air friction. The stone has two different displacements: it can be calculated as the resultant of a steady motion upward and a steady motion downward with a constant acceleration. The function of displacement of this latter one is $s(t) = -\frac{1}{2}gt^2$.)

Equations, inequalities, simultaneous equations

Based on the cuneiform script tables from the ancient Mesopotamia from 2000 BC we know that the scribes of that time could already solve equations and simultaneous equations. According to our knowledge of today on the oldest Egyptian written memory, on the so called Rhind papyrus we can already see the traces of the knowledge of algebra derived from practice and of some exercises leading to equations.

The writer of the papyrus was the royal scribe Ahmes, who came up with calculation problems in connection with everyday life and he also gave their solutions. Later on several similar findings emerged with further problems, indicating the level of development of the calculation techniques of that time, which were used for solving simple equations.



5. Solving equations with elimination, with the “balance method”

While solving the equation it is usually necessary to transform it too. This process is called **rearrangement** of the equation. The transformations, in which we do not lose any of the roots of the equation and we do not get such solutions which are not roots of the original equation (these are called **false roots**), are called **equivalent transformations.**

rearrangement of an equation

equivalent transformations

During the rearrangement we act according to the rules resulting from the **balance method.** It means that during equivalent transformations we do the same operations on both sides of the equation taking the following conditions into consideration:

balance method

During equivalent transformations we do the same operations on both sides of the equation.

- We can **add** or **subtract** the same number to or from both sides of the equation:

$$6x - 3 = 9 \quad /+3,$$
$$6x = 12.$$

- We can **multiply** or **divide** both sides of the equation by the same number different from 0:

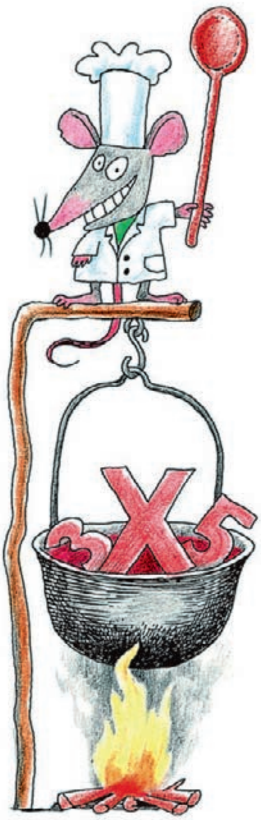
$$6x = 12 \quad /:6,$$
$$x = 2.$$

- We can add or subtract a term containing an unknown if it does not change the fundamental set of the equation:

$$8x = 6x - 2 \quad /-6x,$$
$$2x = -2,$$
$$x = -1.$$

- If we multiply by an expression containing an unknown, then we might get **false roots,** and if we divide by it, then it might lead to **loss of root.** Therefore we preferably should not apply this operation. If it still happens, then we should pay extra attention to the checking of solutions of the equation!

false root, loss of root



Example 1

I thought of a number, I subtracted six from its double, I halved the result, and thus finally I got three. Which number did I think of?

Solution I

We can get the solution of the exercise, if by **elimination** we trace back the resulting numbers in each step.

In the last step when dividing by 2 we got 3 as a result, so the dividend was 6. 6 resulted so that we subtracted 6 from a number, so the minuend was 12. 12 is the double of the number thought of, thus the number though of was 6.

Indeed:

$$\frac{2\cdot 6-6}{2}=\frac{12-6}{2}=\frac{6}{2}=3.$$

Solution II

Let us solve the exercise with an equation. Let the number thought of be x. According to the text of the exercise we can set up the following equation:

$$\frac{2x-6}{2}=3.$$

We can solve this with the help of the **balance method** in the following:

$$\frac{2x-6}{2}=3 \qquad /\cdot 2,$$

$$2x-6=6 \qquad /+6,$$

$$2x=12 \qquad /:2,$$

$$x=6.$$

So the number thought of is 6; we have already made sure of the correctness of it in the previous solution.

Example 2

Let us solve the following equation:

$$3(x-2)-2(-2x+3)+1=\frac{13x+7}{7}-\frac{12x}{14}.$$

Solution

During the solution when doing the transformations our aim is that the equation becomes **simpler** step by step, and that **in the end only the unknown stands on one side.**

On the left side let us open the parentheses, and on the right side let us simplify the second fraction, thus the two fractions will have a common denominator.

$$3x-6+4x-6+1=\frac{13x+7-6x}{7},$$
$$7x-11=\frac{7x+7}{7}.$$

After the combinations and simplifications we arrange the unknowns to the left side of the equation, and we arrange the constant terms to the right side.

constant

$$7x - 11 = x + 1,$$
$$6x = 12,$$
$$x = 2.$$

Based on the checking we can make sure that $x = 2$ is indeed a solution of the equation.

Example 3

Let us solve the following equation:

$$5x-\frac{3}{2x-10}=25-\frac{3}{2x-10}.$$

Solution

The domain of the equation cannot contain the number for which $2x - 10 = 0$, since the fraction does not have a meaning then. It means that every real number can be found in the domain except for 5.

$$2x-10 \neq 0,$$
$$2x \neq 10,$$
$$x \neq 5.$$

We add $\frac{3}{2x-10}$ to both sides of the equation, and thus we get the following equation:

$$5x = 25,$$
$$x = 5.$$

By adding the fraction we added an expression containing an unknown to both sides of the equation, and thus **we changed its domain.** The domain of the resulting equation became the set of the real numbers, and such a solution resulted which cannot be a root of the original equation because of the given conditions. Therefore this result is called a **false root.** The equation does not have a solution.

false root

We get a false root, e.g.:

$$\frac{1}{x}+1=\frac{1}{x} \quad / \cdot x,$$
$$1+x=1,$$
$$x=0.$$

In the original equation x cannot be equal to 0!

Example 4

Let us solve the following.

$$x(3x - 8)(2x + 3) = 6x^3 - 7x^2 + 24.$$

Solution

The domain of the equation: $\mathbb{R}$.

First let us expand the parentheses on the left side, and then let us combine the like terms.

$$\begin{aligned} x(6x^2 + 9x - 16x - 24) &= 6x^3 - 7x^2 + 24, \\ x(6x^2 - 7x - 24) &= 6x^3 - 7x^2 + 24, \\ 6x^3 - 7x^2 - 24x &= 6x^3 - 7x^2 + 24, \\ -24x &= 24, \\ x &= -1. \end{aligned}$$

It can be observed that we did not do any such step during the transformations with which we would have changed the domain of the equation or we would have got new roots. Every step was an **equivalent transformation.** By substitution we can make sure of the correctness of the resulting root.

Exercises

1. Solve the following equations on the set of real numbers.

a) $2(2x + 1) - 1 = 1 - 2(1 + 2x)$; *b)* $\frac{2y}{3} - \frac{3y}{2} = \frac{1}{6}$;

c) $2\frac{1}{3}z - 1\frac{3}{5}(5 - z) = 1$; *d)* $2[2(2v - 1) - 1] - 1 = 0$.

2. Solve the following equations on the set of real numbers.

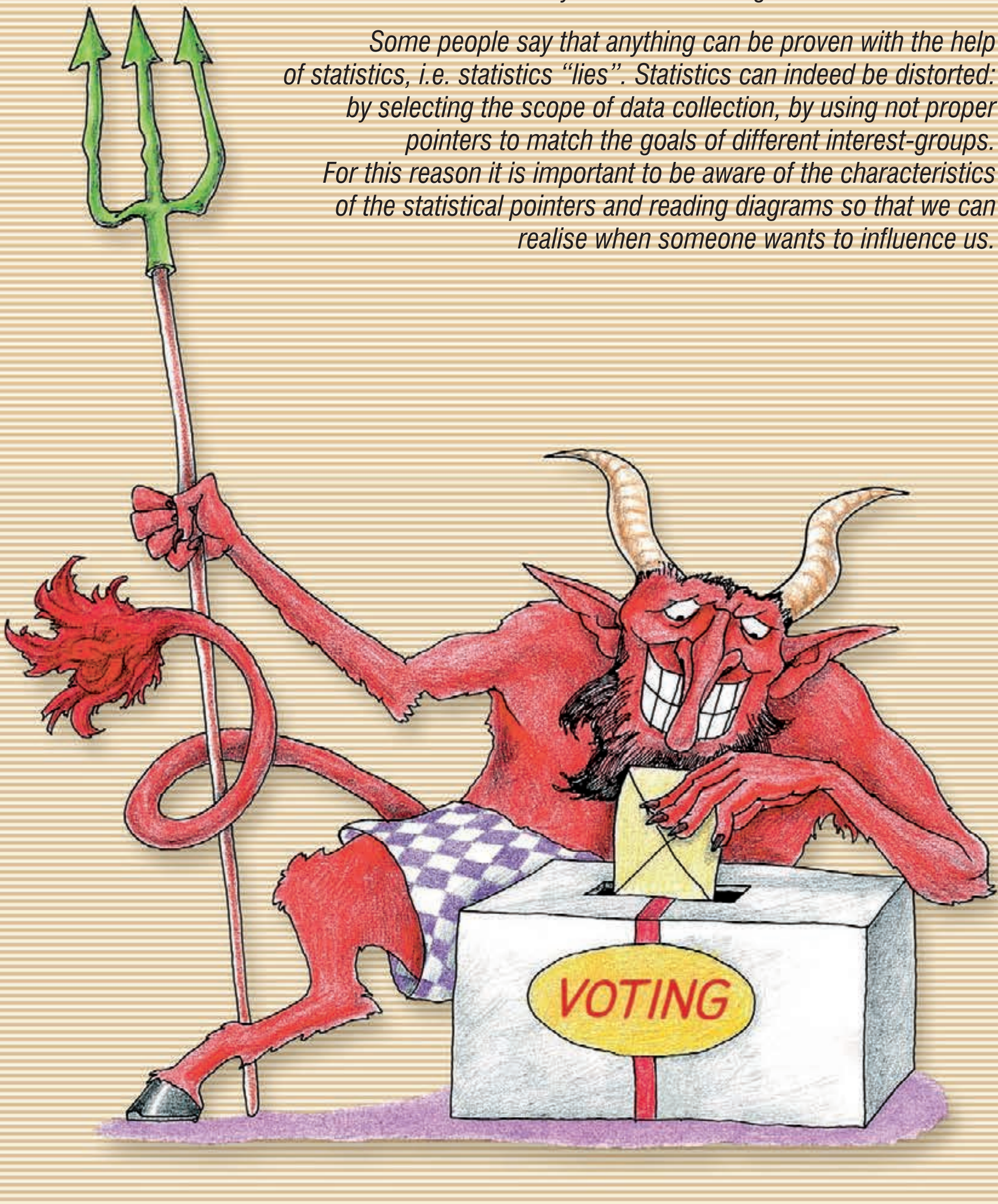
a) $2(x+1)\frac{1}{x} - 1 = 1 - \frac{2(1+2x)}{x}$; *b)* $\frac{2y}{y+1} - \frac{3y}{y+1} = \frac{1}{6}$;

c) $\frac{z}{3z-3} - \frac{5-z}{z-1} = 1$; *d)* $\frac{v-1}{v+1} - \frac{2v-1}{2v+1} = 0$.

Statistics

The science of statistics deals with collecting, organising and evaluating data. Such collection of data is for example an opinion poll, voting, different indicators of the economy, the data of weather, etc. Statistical methods are also used for example when the effect of a newly developed medicine or the efficiency of a new teaching method is examined.

Some people say that anything can be proven with the help of statistics, i.e. statistics "lies". Statistics can indeed be distorted: by selecting the scope of data collection, by using not proper pointers to match the goals of different interest-groups. For this reason it is important to be aware of the characteristics of the statistical pointers and reading diagrams so that we can realise when someone wants to influence us.

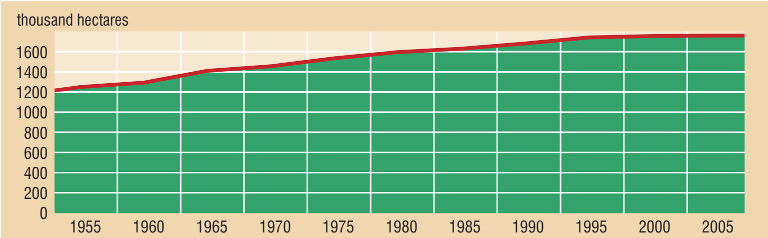


1. The representation of data

The data collected for statistical analysis are usually called the ***set of data*** or ***sample.***

We can collect the data in tables and we can represent them on diagrams. We can come across several such diagrams in the newspapers and in the television.

Year	Area of woods (thousand hectares)
1955	1240
1960	1280
1965	1360
1970	1440
1975	1500
1980	1590
1985	1610
1990	1640
1995	1700
2000	1770
2005	1775



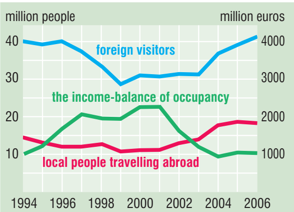
The changes in the area of woods in a country

Figure 1

For example the table and the diagram in figure 1 show the changes in the area of woods in a country from 1955 to 2005.

We can follow the changes of and can compare the different data well if we represent them in the coordinate system with the help of ***curves.*** ***(Figure 2)***

In many cases the information read from the diagrams could only be communicated with several tables or text. An image is not only more demonstrative but also provides more information.

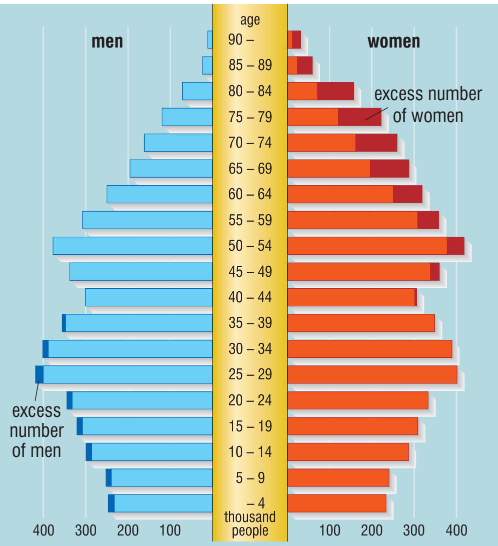


The trend and the income-balance of occupancy

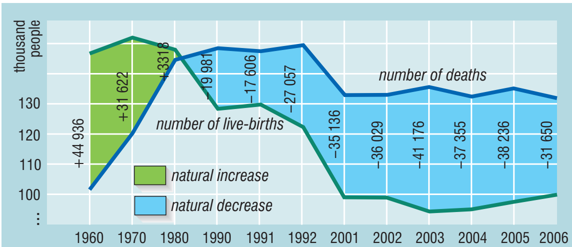
Figure 2

The following ***column charts*** show the changes in the population of a country in Central Europe in certain age groups ***(Figure 3)***. The curves, column charts are useful when we are interested in the changes of or in the relation of the data.

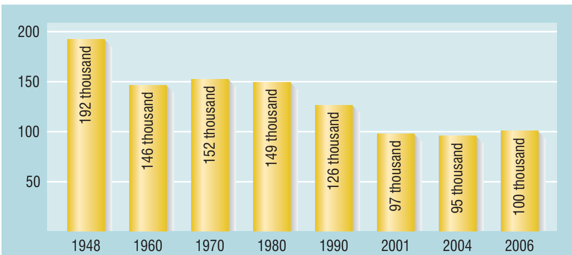
Figure 3



The age pyramid of the population of a country in Central Europe (2006)



The changes in the natural increase (or decrease)



The changes in the number of new-borns

Example 1

Albert collected 20 points in his Mathematics test, while Bruce collected 25 points in his Physics test. Who has the better test result if the total points were 25 for the Mathematics test and 50 for the Physics test?

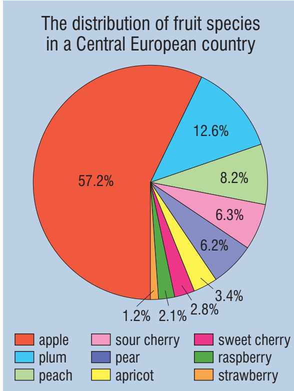


Figure 4

Solution

Although Bruce collected more points than Albert, it is only 50% of the total, while Albert's result is 80% of the total, so Albert has the better test result.

It often happens that comparing the amounts measured in different ways does not provide the adequate information; in cases like this **comparing the percentages or ratios** shows a more realistic picture.

pie charts

If we are interested in the proportion of data compared to the whole, then it is worth using a ***pie chart*** (or a three-dimensional pie chart) to represent them, where the circle is divided in the ratio of the corresponding data ***(Figure 4)***.

Figure 5 and 6 show well when it is more helpful to use a column chart and when to use a pie chart (or a three-dimensional pie chart):

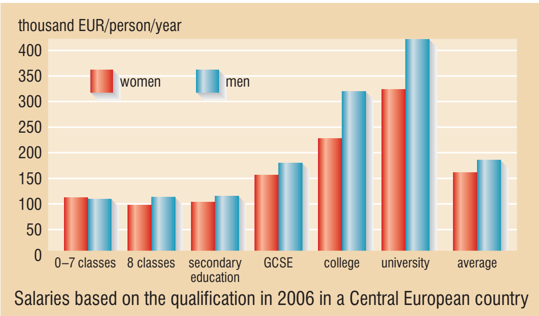


Figure 5

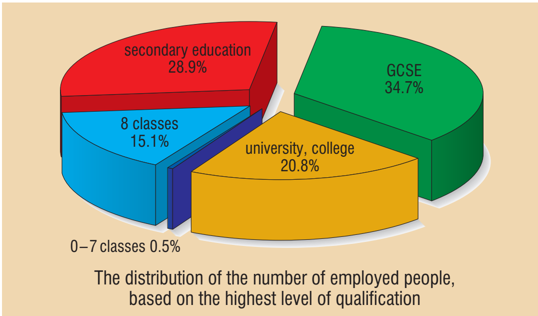


Figure 6

frequency

histogram

frequency distribution

The number of occurrence of each data is shown by the ***frequency*** which can be represented on a ***frequency diagram,*** in other word on a ***histogram.***

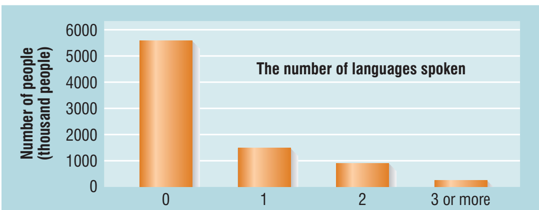
The data and their frequency together constitute a ***frequency distribution.***

We often classify data; we count how much data there is in each class and thus we get the frequency of classes.

The result of the survey about the language knowledge of citizens older than 14 in a Central European country is as follows:

The number of languages spoken	Number of people
Does not speak a foreign language	5603 thousand
Speaks 1 foreign language	1483 thousand
Speaks 2 foreign languages	906 thousand
Speaks 3 or more foreign languages	247 thousand

Figure 7



relative frequency

If we give the data in the ratio of the whole population older than 14, i.e. we calculate what portion, what percentage of the people older than 14 speak 0, 1, 2, 3 or more languages, then we get the ***relative frequency*** of the data, which often makes a better comparison of the data possible. It can be represented on a histogram too.

Example 2

A class wrote a test in Mathematics. The maximum attainable score was 50 points. The scores the students got:

18; 22; 37; 42; 48; 50; 32; 38; 26; 40;
42; 43; 45; 35; 34; 36; 39; 40; 34; 33.

The teacher followed this pattern for the grades:

A: 43–50; B: 36–42; C: 29–35; D: 22–28; F: 0–21.

Give the frequency distribution of the grades and represent them on a column chart.

Solution

The frequency distribution is shown in figure 9; the column chart is shown in figure 8.

Figure 8

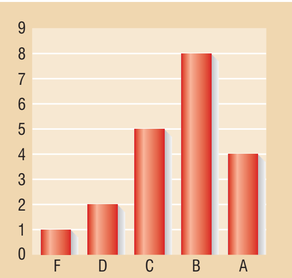


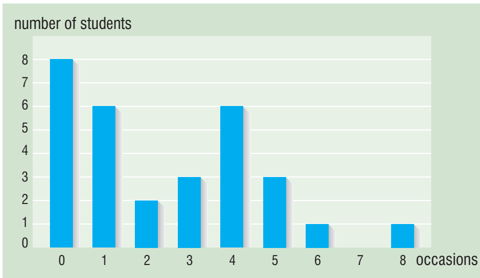
Figure 9

Grade	F	D	C	B	A
Frequency	1	2	5	8	4

Exercises

1. A survey was conducted in a class regarding how many times the students had gone to the theatre in the past year. The result is shown in the column chart.

Read and tabulate the frequency distribution of the data.



2. Represent the area of the continents and the population on a column chart.

Continent	**Area** (1000 km^2)	**Population** (million people)
Europe	10 508	732
Asia	44 411	3969
Africa	30 319	924
North America	21 515	332
Central and South America	20 566	566
Australia and Oceania	8 510	34
Antarctica	13 328	–
Total of the world	149 157	6555

3. In the table the data of the regions of Hungary is shown. Create a diagram based on the data.

	Central Hungary	Central Transdanubia	Western Transdanubia	Southern Transdanubia	Northern Hungary	Northern Great Plain	Southern Great Plain
Area (%)	7.4	12.1	12.0	15.2	14.4	19.1	19.8
Population (%)	28.3	11.0	9.8	9.7	12.7	15.1	13.4
Gross national product (GNP) (%)	41.6	10.0	10.3	7.8	8.8	10.6	10.9
Unemployed people (%)	15.2	10.5	6.9	11.8	19.2	22.2	14.2
Foreign capital investment (%)	64.0	6.8	8.9	3.3	7.7	4.7	4.6
Investments (%)	39.4	12.8	11.3	6.9	9.5	11.4	8.7

4. The club is buying basketball shoes for the members of a girls basketball team. The length of the foot of the girls was measured in centimetres accurate to the millimetres, and the following results were obtained:

23.2; 26.3; 28.0; 25.1; 25.8; 24.9; 24.2; 25.4; 25.9; 26.1; 24.4; 24.9; 23.6; 23.4.

Give the frequency distribution of each size and represent them on a column chart, if the size conversion table of basketball shoes is as follows:

European size	36.0	36.5	37.5	38.0	38.5	39.0	40.0	40.5	41.0	42.0	42.5	43.0
Length of foot (cm)	22.5	23.0	23.5	24.0	24.5	25.0	25.5	26.0	26.5	27.0	27.5	28.0

5. Observe how many hours you spend a day with the following activities: sleeping, school, homework, entertainment, eating and other. Represent the data on a column chart and on a pie chart.

Collect data on the number of students of the class who on average sleep less than 4 hours, 4 to 6 hours, 6 to 8 hours, more than 8 hours a day. Represent the frequency distribution on a column chart.

6. Collect and represent data on a histogram regarding how many students from the class go to school on foot, by bicycle, by public transport or by car.

7. Create a frequency table showing how many times each letter occurs on the first page of the first lesson of your history book. Represent it on a histogram. Based on this, determine which letters are the most common in your language? In one of the games one should figure out a word and can guess 6 letters. The letter which appears in the word will be shown. Which letters would you guess? Why is the chance for hitting small?

8. Experiment with two coins. Flip the two coins at once 30 times in a row and take notes on how many times 0 head, 1 head or 2 heads appear on the two coins. Represent it on a column chart. Choose the value appearing most often.

Puzzle

In a quiz show one answer should be chosen out of the four possible answers. The audience is allowed to give assistance to the player. The question was as follows: "Whose comedy is musical My Fair Lady based on?"

A) **Shakespeare** ***B)*** **Oscar Wilde**
C) **G. B. Shaw** ***D)*** **Neil Simon**

In the diagram of the votes the columns show the votes given for answers *A*, *B*, *C*, *D* respectively. Which answer shall we choose?